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Fairly Amenable Semigroups

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Abstract

Amenability developed alongside modern analysis, as it is a central property lacking in a group used to show, for example, the Banach-Tarski paradox (Wagon, 1993). The first working definition was given by von Neumann (1929), in terms of finitely-additive measures. A number of useful theorems are capable of being shown using this basic definition.

The first modern definition of amenability was given by M. M. Day (1957), whose concept involved invariant means. For groups this coincides exactly with the von Neumann condition: each invariant mean corresponds to an invariant finitely-additive measure, corresponding via Lebesgue integration. This advance was significant as it opened the door to the application of abstract harmonic analysis, fixed-point theorems, and an industry of consequences. Amenable groups support almost-invariant finite means, and via decomposition this is culminated as the Følner condition, a statement about finite sets. Abelian groups are amenable as a simple consequence of the Markov-Kakutani fixed-point theorem. A theorem of B. E. Johnson (1972) led to the development of amenable Banach algebras and C*-algebras, neatly encoding amenability in the mechanics of cohomology theory.

While amenability is directly generalisable from groups to semigroups, the two key definitions do not correspond in the same way as they do for groups: extracting a finitely-additive measure from a left-invariant mean yields what might be called a left *preimage-invariant* measure, and for groups these merely correspond to the inverse elements. A simple but surprising consequence of Day's definition of amenability is that semigroups with a zero element are both left and right amenable (Day, 1957). Yet they cannot support a (totally) invariant finitely-additive measure (van Douwen, 1992, p231). On the other hand, all semigroups with more than one distinct left zero are not left amenable (Paterson, 1988), and in particular there are many non-amenable finite semigroups, which is another contrast to the group case: all finite groups are amenable. This standard definition of amenability for semigroups is therefore unintuitive and, perhaps, unsatisfactory. Restricting to better-behaved classes of semigroups, such as the inverse semigroups, does little to improve this.

The first new result of the present work is that there is a weakening of invariance that can be used in the context of finitely-additive measures to generalise group amenability to semigroups in a different way. For a semigroup S, a finitely-additive measure $\mu \in [0,1]^{\mathcal{P}(S)}$ will be called left *fairly invariant* if, for all $s \in S$ and $A \subseteq S$ such that $\lambda_s|_A$ is an injection, $\mu(sA) = \mu(A)$. When a semigroup supports such a finitely-additive measure, then it is left *fairly amenable*. Fair amenability is a generalisation of group amenability, and retains some of the useful theorems. Some of the results shown using this formulation include: a semigroup is left fairly amenable when it satisfies a weakened Strong Følner Condition, finite semigroups are all fairly amenable, semigroups with involution are either fairly amenable on both the left and the right or not at all, adjoining a zero does not cause a non-fairly amenable semigroup to become fairly amenable, directed unions of fairly amenable semigroups are fairly amenable, and a variety of examples which are fairly amenable or not fairly amenable.

The name "amenable" is, as the story goes, supposed to be a pun, since a*men*able groups support invariant *means*. Thus an important question for fair amenability is, what condition for a mean is equivalent to the fair invariance of the corresponding finitely-additive measure? One approach is to flip the duality between the convolution action in $\ell^1(S)$ and the dual action in $\ell^\infty(S)$ upside-down: attempt convolution in $\ell^\infty(S)$ and the dual action in $\ell^1(S)$. In this scenario, the curious will consider such ill-defined expressions as $0 * \chi_S$. Fortunately, wherever the convolution *partial* action of s on $\phi \in \ell^\infty(S)$, i.e. $s * \phi$, is well-defined and bounded, then the integral with respect to a left fairly-invariant measure can be readily computed. It is shown that a semigroup S left fairly amenable if, and only if, there exists a mean m such that $m(\phi) = m(s * \phi)$ for all $s \in S$ and $\phi \in \ell^\infty(S)$ such that $s * \phi \in \ell^\infty(S)$. Hence the nomenclature "fairly amenable" is justified as a pun also.

Some variations on fair amenability and related results are also explored. As a variation on the $*$ partial action, an operator $\circledast$ is introduced on $\ell^\infty(S)$, which induces a full action of S. One drawback of $\circledast$ compared to $*$ is that, in order to express fair amenability, an additional condition is required to limit the scope of invariance appropriately. Finally, inner $\circledast$ invariance and its "fair" variant are briefly explored.

Declarations

Declaration of Originality

This Thesis contains no material which has been accepted for a degree or diploma by the University or any other institution, except by way of background information and duly acknowledged in the Thesis, and to the best of my knowledge and belief no material previously published or written by another person except where due acknowledgement is made in the text of the Thesis, nor does the Thesis contain any material that infringes copyright.

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Chapter 5 has been sent in a reduced form for potential publication in the journal Semigroup Forum, though at this writing it has not yet been accepted. At such time that a journal holds the copyright for content, access to the material should be sought from them in accordance with their policies.

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Joshua Thomas Deprez Date

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In a way, this work is the culmination of much more than just three-and-a-half years of reasearch, but additionally many years of schooling and undergraduate study as well. It is a testament to the quality of the public education system. I would like to thank all the motivating and engaging teachers I have had over the years. Finally, to anyone who suggested I could have done better if I had a private school background instead: I wouldn't have changed a thing.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 What is amenability?

Unwisely, I began seeking answers to this question in a pale blue book titled "Amenability" by Paterson (1988). I write "unwise" (perhaps unfairly to Paterson as he was perhaps aiming at a higher audience than myself), but perhaps merely unwise for me, as within a few pages of the introduction I realised I had been bamboozled with a sufficiently broad array of vaguely familiar background concepts to render the full notion of amenability unclear. It seemed it all had something to do with functionals, Lebesgue integration, measures, groups and semigroups and topologies, the Haar measure, the Borel subsets, function spaces, Banach algebras, C*-algebras, von Neumann algebras, virtual and approximate diagonals, cohomology, injective and projective tensor products, strong and weak convergence, nuclearity…But just *what is it*?

The heart and soul of amenability is the idea of *invariance of content*. By "content", we usually mean more tangible concepts such as volume and mass. Invariance is a most natural concept to consider, as it appeals to the intuition in a variety of ways. For example, we know from real-world experience that merely moving an ordinary tangible, physical object does not alter its mass or its volume. Only when the object undergoes certain deformations do these quantities change. Thus, we can say that the content of a tangible object is invariant under rigid motion: translation and rotation.

The Banach-Tarski paradox, and its precursors such as the Hausdorff paradox, showed that our usual formalisations for the ideas of volume, translation and rotation have very non-intuitive consequences. This paradox was a very forceful demonstration, since it takes place within $\mathbb{R}^3$, which we intuitively consider to be a reasonable approximation of the tangible universe, and uses rotations and translations, which we intuitively consider to be volume- and mass-preserving operations. The conclusion of the Banach-Tarski paradox is that any bounded non-empty set inside $\mathbb{R}^3$ can be decomposed into subsets, and these subset "pieces" translated and rotated, and recombined to form *any other bounded set inside* $\mathbb{R}^3$. An image to hold in mind that is often described at this point is of a pea split up and reassembled into a ball the size of the sun!

It was known that paradoxes of this nature only occurred in spaces of dimension 3 or higher, which indicated that perhaps it was some fault of $\mathbb{R}^3$, or the fault of the group G_3 which models the group of rotations and translations of points in $\mathbb{R}^3$. This is true, but obscures the real problem. von Neumann (1929), who presented the first definition of amenability, noted that the Banach-Tarski paradox is an issue of a group-theoretic nature, and was not limited to either $\mathbb{R}^3$ or G_3. The group that is genuinely central to the paradox, the *free group on two generators* denoted $\mathbb{F}_2$, simply has a structure that circumvents any reasonable definition of invariant volume. Thus $\mathbb{F}_2$ is said to be *paradoxical*. Its embedding in G_3 extends the problem into the form of the Banach-Tarski paradox. It then happens that $\mathbb{F}_2$ is not amenable because it is paradoxical, and *vice-versa*.

There is now a multitude of definitions for amenability. For the case of *groups*, all the definitions effectively coincide and follow from each other, which is why they may all be considered to be definitional conditions. This concord provides a useful array of tools for determining the property of amenability for a given group, as well as a broad set of nice analytic properties that hold for the amenable groups. These varying definitions of amenability are introduced in Chapter 2.

On semigroups, however, it simply doesn't work that way. The most popular definition of amenability originated with Day (1957), as a natural-looking generalisation of his previous work on the group case. For groups, there is a correspondence between the *means* (a certain kind of functional) and the finitely-additive measures which were the core of von Neumann's definition. On semigroups, however, this important correspondence breaks down almost immediately and with little fanfare, and this is where my confusion really began.

1.2 Preliminaries

1.2.1 A remark about proofs

Many of the proofs contained in this thesis follow a presentation style that aims to improve validity. In these I have aimed to emulate the *hierarchical structure* advocated by Lamport (2012). These proofs may seem unusual, and would appear out of place in journals that would prescribe a particular formatting. While it was tempting to stick to the safety of a more classical, conventional style, I reasoned that there would be no better place to give the concept a run than a thesis containing both standard results, and original results.

1.2.2 Semigroups, groups, self-actions

Recall the following basic definitions.

Definition 1.1 A *semigroup* is a set S together with a binary operation $\cdot : S \times S \to S$, usually denoted with concatenation, such that $\cdot$ is associative, i.e.

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \text{for all } a, b, c \in S.$$

Usually S stands for both the semigroup and the underlying set. An identity element 1 is such that $1x = x1 = x$ for all $x \in S$. We shall write S^1 to mean S *with an identity adjoined if necessary*. If $S = S^1$ then S is a *monoid*. Similarly, a zero element 0 satisfies $0x = x0 = 0$ for all $x \in S$, and again similarly, write S^0 to mean S *with a zero adjoined if necessary*. Note that this should not be confused with semigroups where 0 could be an additive identity: applying the previous statements strictly, $\mathbb{N}^0$ is the semigroup of positive natural numbers with a zero element adjoined, and with regards to addition, that zero behaves like $-\infty$. On the other hand, $\mathbb{N} \cup \{0\}$ is the Abelian monoid of natural numbers with 0 being the identity element.

If $A \subseteq S$ and $s \in S$, then $sA := \{sa : a \in A\}$ and $As := \{as : a \in A\}$. This is a special case of the product of sets: for any $A, B \subseteq S$, $AB := \{ab : a \in A, b \in B\}$.

A *group* G is a semigroup such that $xG = Gx = G$ for all $x \in G$. It is easy to show that this is equivalent to G being a semigroup with an identity element and inverses, in the fashion usually taught in undergraduate studies. Inverse elements in groups are generally denoted with the superscript $^{-1}$, and for a set $A \subseteq G$, the notation A^{-1} shall mean the set consisting of the inverse of each element of A, i.e.

$$A^{-1} := \left\{a^{-1} : a \in A\right\}.$$

For a set X, the semigroup $\mathcal{T}_X$ denotes the *transformation semigroup* of X, consisting of all maps $f : X \to X$ with respect to composition of functions. A special and important subgroup of $\mathcal{T}_X$ is Sym_X, the *symmetric group* of X, consisting of all bijections from X to itself with composition.

Definition/Theorem 1.2 Let S be a semigroup, and define the maps

$$\lambda_s(x) := sx, \quad \rho_s(x) := xs \quad \text{for all } s, x \in S.$$

The maps $\lambda : s \mapsto \lambda_s$ and $\rho : s \mapsto \rho_s$ are known as the *left regular* and *right regular* representations, respectively. For all $s \in S$, λ_s and ρ_s are elements of $\mathcal{T}_S$, and so λ and ρ are homomorphisms from S to $\mathbb{T}_S$, though they are not necessarily faithful. However, these representations can be used to demonstrate such results as the following.

- The semigroup S is faithfully embedded within some transformation semigroup, in particular, $\mathcal{T}_{S^1}$.
- If S is a group, then it is faithfully embedded in thhe symmetric group Sym_S, and this result is known as Cayley's theorem.
- If S is an inverse semigroup, then it is faithfully embedded in the symmetric inverse monoid $\mathcal{I}_X$, i.e. the Wagner-Preston theorem.

(Not shown here.) □

Thus it is always natural to imagine a group as *symmetries* (bijections), a semigroup as *transformations*, an inverse semigroup as *partial* symmetries. The representations λ, ρ provide examples of *self-actions*—actions of S on itself. These may be extended to actions of S on the power set $\mathcal{P}(S)$, i.e. $\lambda_s(A) = sA$ for $A \subseteq S$. We may also speak of the preimage under λ_s of a set A, being $\lambda_s^{-1}(A) = \{t : \lambda_s(t) \in A\} = \{t : st \in A\}$. For notational convenience, I shall use, wherever it is not confusing to do so, the shorthand

$$s^{-1}A := \{t : st \in A\}, \quad As^{-1} := \{t : ts \in A\}$$

for all $s \in S$ and $A \subseteq S$. If S is a group then $s^{-1}A$ and As^{-1} are unambiguous, since in that instance $\left\{s^{-1}t : t \in A\right\} = \{t : st \in A\}$, similarly on the right.

1.2.3 Function spaces

While this work is not aimed at providing a comprehensive account of modern analysis and representation theory, it will be useful to recall some salient ideas. For a thorough account of Banach spaces, see Allan (2011). For understanding amenability the basic definitions require, at minimum, no regard to topology.

Definition 1.3 Let S be a set, and consider the functions from S to $\mathbb{K}$ (denoted $\mathbb{K}^S$. $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$). For $f, g \in \mathbb{K}^S$ the addition $f + g$ defined pointwise is an Abelian group, and functions can be scaled by any scalar $\lambda \in \mathbb{K}$, so, $\mathbb{K}^S$ is a *vector space* in which the functions are the vectors.

Recall the following important subspaces of $\mathbb{K}^S$. Let $f \in \mathbb{K}^S$.

- $\ell^1(S)$ is the space of absolutely summable functions on S:

$$f \in \ell^1(S) \quad \Leftrightarrow \quad \|f\|_1 := \sum_{s \in S} |f(s)| < \infty.$$

- More generally, $\ell^p(S)$ (for $0 < p < \infty$) is the space of p-th power summable functions:

$$f \in \ell^p(S) \quad \Leftrightarrow \quad \|f\|_p := \left[\sum_{s \in S} |f(s)|^p\right]^{1/p} < \infty.$$

- $\ell^\infty(S)$ is the space of bounded functions:

$$f \in \ell^\infty(S) \quad \Leftrightarrow \quad \|f\|_\infty := \sup_{s \in S} |f(s)| < \infty.$$

It is a straightforward exercise to demonstrate that these are sub*spaces* of $\mathbb{K}^S$, that $\|\cdot\|_1, \|\cdot\|_p, \|\cdot\|_\infty$ are all *norms*, that $\ell^1(S), \ell^p(S), \ell^\infty(S)$ are norm-complete (*Banach spaces*), and that when S is finite they all coincide with $\mathbb{K}^S$ and all three norms are equivalent. A special case is $p = 2$: $\ell^2(S)$ is a *Hilbert* space having the *inner product* $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle := \sum_{s \in S} f(s)g(s).$$

Again, it is straightforward to verify this is an inner product and that it yields the norm $\|\cdot\|_2$ (given by $\|x\|_2 = \langle x, x\rangle^{1/2}$). Usefulness of the "inner product" given above extends beyond $\ell^2(S)$, and so, if outside the context of any other given Hilbert space, we shall leave $\langle\cdot,\cdot\rangle$ defined as above wherever the right-hand side converges.

If S is a semigroup, then the semigroup structure can interact with the elements of these function spaces in a variety of ways. For $s \in S$, one common approach is to multiply on the left or right by s before applying $f \in \mathbb{K}^S$, and this is called the *(dual) action of* s *on* f:

$$(sf)(x) := f(sx), \quad (fs)(x) := f(xs).$$

One must have a grasp of the space of *bounded linear operators* $\mathcal{B}(E,F)$ consisting of linear maps $T : E \to F$ satisfying

$$\|T\| := \sup_{x\neq 0} \frac{\|Tx\|_F}{\|x\|_E} < \infty,$$

and the special case of the *dual* of a space E, denoted E^*, where $E^* := \mathcal{B}(E,\mathbb{K})$. An example of a functional $\Sigma \in \ell^1(S)^*$ is

$$\Sigma(f) := \sum_{s\in S} f(s) \quad \text{for all } f \in \ell^1(S).$$

Isometric isomorphism is the standard for considering when two spaces are to be regarded as equivalent."Taking the dual" is then sometimes involutive ($\ell^p(S)^{**} \equiv \ell^p(S)$ if $1 < p < \infty$) and sometimes not (e.g. $\ell^1(S)^{**}$ is not isometrically isomorphic to $\ell^1(S)$), but that is not the most that can be said.

Theorem 1.4 The dual space $\ell^1(S)^*$ can be, and almost always is, identified with $\ell^\infty(S)$, and $\ell^1(S)$ is *canonically* isometrically embedded in $\ell^1(S)^{**} \equiv \ell^\infty(S)^*$.

Proof (sketch) For each $\phi \in \ell^\infty(S)$ define the corresponding $\hat{\phi} \in \ell^1(S)^*$ by

$$\hat{\phi}(f) := \sum_{s\in S} f(s)\,\phi(s) = \langle \phi, f\rangle.$$

This implies that for a given $\hat{\phi}$ there exists a ϕ given by $\phi(s) = \hat{\phi}(\delta_s)$. Similarly, for each $f \in \ell^1(S)$ define the corresponding $\hat{f} \in \ell^1(S)^{**}$ by

$$\hat{f}(\phi) = \phi(f) \quad \text{for all } \phi \in \ell^1(S)^*.$$

This works for embedding any linear space E in E^{**}. Note that there may exist $\hat{f} \in \ell^1(S)^{**}$ for which there is no corresponding f. It is then a matter of verifying the isometric isomorphism. $\square$

For any two real-valued functions f and g, let the condition $g \leq f$ be assumed to be taken pointwise, i.e. $g(t) \leq f(t)$ for all t.

Definition 1.5 Given an $f \in \ell^\infty(S)$ such that $f \geq 0$, if there exists a finite index set I, a collection $\{A_i\}_{i \in I}$ of subsets of S, and collection $\{a_i\}_{i \in I}$ of positive real numbers such that

$$f = \sum_{i \in I} a_i \chi_{A_i},$$

then f is a (n-step) *simple* function. Note that f takes at most n distinct values in its range. The collection of sets $\{A_i\}_{i \in I}$ may be assumed, without loss of generality, to be pairwise disjoint, since any simple function can be re-expressed with pairwise disjoint sets: use $I = f(S)$, and for each $i \in I$ use $A_i = f^{-1}(\{i\})$ and a_i be the value in the singleton set $f(A_i)$.

1.2.4 Ultrafilters and ultralimits

Finally, a useful tool for resolving some problems is the ultralimit. Even in seemingly benign situations, ordinary sequential limits can fail to exist, and one solution is to take an ultralimit instead. Ultralimits are closely related to non-standard analysis, and a good introduction was posted by Tao (2010). Certain unnamed mathematicians may consider ultralimits to be the "fairy godmothers of mathematics," but the basic ultralimit construction described below is simultaneously useful and devoid of much mystery.

Definition 1.6 A *filter* F on a set X is a collection of subsets of X satisfying the following properties:

(F1) $\emptyset \notin F$ (this mostly serves to exclude the "improper" filter $\mathcal{P}(X)$).

(F2) If $A \in F$ then every superset B is also in F (it is *upwards-closed*).

(F3) If A and B are in F, then $A \cap B \in F$.

Note that a filter F cannot contain any pair of disjoint sets—by (F3) their intersection would be in F, but by (F1), this is a contradiction.

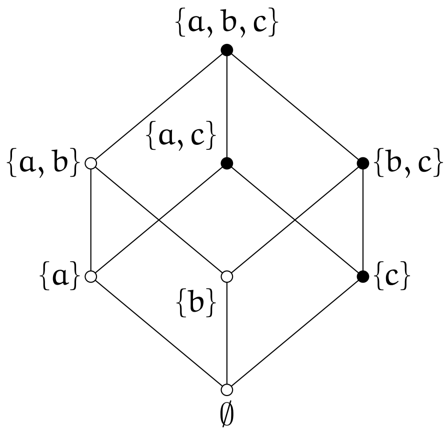

Figure 1.1: An ultrafilter (filled points) on the finite set $\{a, b, c\}$.

If a set $A \notin F$ then every subset is also not in F. To see this, suppose some subset $B \subseteq A$ is in F: by (F2), this implies that $A \in F$, contradicting $A \notin F$.

Definition 1.7 An *ultrafilter* U is a filter that satisfies the further property that for any $A \subseteq X$, either A or $X \backslash A$ is in U.

Since A and $X \backslash A$ are disjoint, by the above, precisely one of them is in any ultrafilter on X. It is not possible to include more sets in an ultrafilter without it failing to be a filter, so there is no filter *finer* than an ultrafilter.

Lemma 1.8 There are two disjoint kinds of ultrafilters on a set X: the *principal* ultrafilters, containing exactly one singleton set $\{x\}$ where $x \in X$, and the *non-principal* or *free* ultrafilters, which contain no finite sets at all.

Proof If an ultrafilter U on X contains any finite set F, we may see that it necessarily contains a singleton set, and is therfore principal, as follows. Assume that $\{x\} \notin U$ for all $x \in X$. In particular $\{f\} \notin U$ for each $f \in F$. Thus $X \backslash \{f\} \in U$ for all $f \in F$, and by property (F3) we thus have $F \cap (X \backslash \{f\}) = F \backslash \{f\} \in U$ for all $f \in F$. Inductively each element of F can be removed, one at a time, until none remain, at which point it contradicts (F1). $\square$

While it is easy to exhibit a principal ultrafilter on many infinite sets, one must rely on some additional axiom to provide the existence of free ultrafilters. The *Ultrafilter Lemma* (UL) does this nearly directly: UL asserts that every filter is contained within an ultrafilter, and some filters, such as the Fréchet filter described below, can only be contained within a free ultrafilter. The Ultrafilter Lemma is also a consequence of the Axiom of Choice. However, the Axiom of Choice does not follow from the Ultrafilter Lemma, so UL is a weakening of AC. One useful consequence of the Ultrafilter Lemma is the Hahn-Banach theorem, but again, the converse does not hold.

Since free ultrafilters do not contain any finite sets, by definition they must contain all cofinite sets. The filter consisting of cofinite sets, the *Fréchet filter*, is therefore contained in any free ultrafilter. Conversely, any ultrafilter containing the Fréchet filter is a free ultrafilter. The Fréchet filter itself is not usually an ultrafilter: consider two infinite sets which are complements, and so neither is cofinite.

Definition 1.9 Given any sequence of real or complex numbers $\{x_n\}$ and a free ultrafilter U over $\mathbb{N}$, if for some value x and each $\epsilon > 0$, we have

$$\{n : |x_n - x| \leq \epsilon\} \in U,$$

then x is the (unique) *ultralimit* of the sequence with respect to U, written

$$x = \lim_U x_n \quad \text{or} \quad x = U\text{-}\lim x_n.$$

This generalises to sequences in any metric space (X, d) by

$$x = \lim_U x_n \quad \Leftrightarrow \quad \{n : d(x_n, x) \leq \epsilon\} \in U.$$

Note that if we try to do the same thing with a principal ultrafilter, then we allow the ultralimit to take the value of the sequence at a single point. For the ultralimit and ordinary limits to coincide, one must require a free ultrafilter.

Some authors sometimes require, for ultralimits, that the ultrafilter U over $\mathbb{N}$ used, instead of explicitly being required to be "free", is required to contain every interval $[n, \infty)$ for each $n \in \mathbb{N}$.

Lemma 1.10 An ultrafilter U on $\mathbb{N}$ is free if, and only if, it contains every interval of the form $[n, \infty)$ for each $n \in \mathbb{N}$.

Proof A free ultrafilter contains all cofinite sets, including as examples the intervals of the form $[n, \infty)$.

To show the converse, suppose an ultrafilter U contains every interval $[n, \infty)$ for $n \in \mathbb{N}$, and assume that U is principal. U, being principal, contains a singleton set,

$\{k\}$ say, and therefore does not contain $\mathbb{N}\backslash\{k\} = [1, k) \cup [k+1, \infty)$. Since subsets of sets not in a filter are also not in the filter, this implies $[k+1, \infty)$ is not in U. This contradicts the requirement that every $[n, \infty) \in U$. $\square$

Ultralimits provide a resolution to the problem of defining a unique limit for a sequence that fails to converge due to having multiple limit points, effectively by choosing one of the limit points. Intuitively, a free ultrafilter will contain a set of indexes corresponding to one of the convergent subsequences, but none of the others, as they will necessarily be indexed by sets in the complement of the index set of the chosen subsequence. If a sequence is bounded, then by the Bolzano-Weierstrass theorem there is at least one convergent subsequence, so an ultralimit is always defined on bounded sequences.

But not only do ultralimits assign a value for the limit of a not-necessarily-convergent sequence, they do so in a completely consistent way when the same free ultrafilter is used.

Lemma 1.11 Let $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ be two sequences and U a free ultrafilter over $\mathbb{N}$. Then

$$\lim_U (x_n + y_n) = \lim_U x_n + \lim_U y_n$$

where the ultralimits exist.

Justification This is a specific case of the *transfer theorem*, stating all operations involving finitely many standard objects can be carried over into the non-standard universe (of ultralimit objects), which is a special case of the formalisation known as Łoś's theorem. (Tao, 2010) $\square$

Chapter 2

Amenability and Groups

The core idea of amenability is that of an invariant structure: some set of quantities that remains unchanged under some action. The first coherent conception of amenability was as a condition on a group that precludes the group generating a Banach-Tarski-style paradox.

2.1 Finitely-additive measures

The first definition follows Definition 10.1 from Wagon (1993). Despite its lengthy pedigree (von Neumann, 1929), it endures as a useful definition (for example, Akhmedov et al., 2009).

Definition 2.1 (Amenability–1) Let G be any group, and $\mu : \mathcal{P}(G) \to \mathbb{R}^{+}$. We say

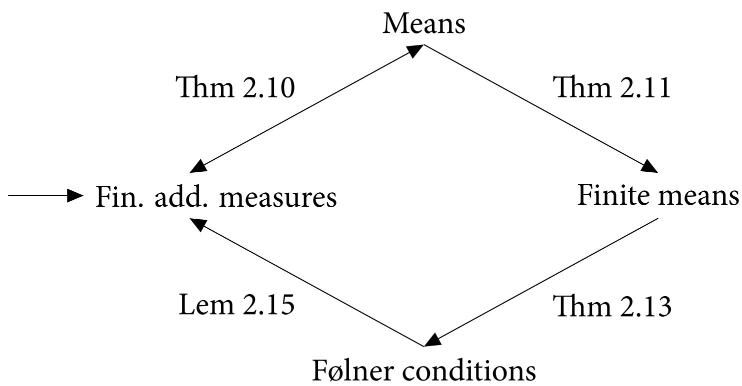

Figure 2.1: The journey of the first half of this chapter.

that μ is

(i) *finitely-additive* if for all disjoint subsets $A, B \subseteq S$, $\mu(A \cup B) = \mu(A) + \mu(B)$,

(ii) *left-invariant* if $\mu(gA) = \mu(A)$ for all $A \subseteq G, g \in G$. similarly, μ is *right-invariant* if $\mu(Ag) = \mu(A)$, and

(iii) a *probability measure* if $\mu(G) = 1$. The *total measure* of μ is the quantity $\mu(G)$.

We say the group G is *left amenable* if there exists a non-negative finitely-additive left-invariant probability measure μ. Similarly, G is *right amenable* where there exists a μ that is right-invariant. If G is neither, it is *non-amenable*.

Put more intuitively, if elements of a group represent motions or rotations, and subsets of the group are "pieces" that can be moved or rotated (by the usual self-action), then amenability guarantees at least one notion of "area" or "volume" of pieces that is invariant with respect to whatever motions a piece is subjected to. Finite additivity ensures that the measure captures the notion of the "fraction of space" the piece takes up within the whole group. Expecting invariance for the action of every element is compatible with the notion of an action being a bijection, with bijections preserving set cardinality.

We require mere *finite* additivity instead of countable additivity because for infinite groups it is easy to show that such a measure cannot exist (Wagon, 1993).

Some basic facts about (not necessarily invariant) finitely-additive measures:

- If $A \subseteq B$ then $\mu(B) = \mu(A) + \mu(B \backslash A)$, and since $\mu \geq 0$, it follows that $\mu(A) \leq \mu(B)$, i.e. μ is *monotone*. Obviously $\mu(A) \leq \mu(B)$ does not necessarily imply $A \subseteq B$.
- For any sets A, B, $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$, i.e. μ satisfies the principles of inclusion-exclusion, similar to set cardinality.
- Since $\mu(G) = 1$, $\mu(\emptyset) = 0$.

Finally, ready examples of finitely-additive measures (but that aren't necessarily invariant in any way) on a set X is built from any ultrafilter on X. Given an ultrafilter $\mathcal{U}$ on X, simply set

$$\mu(A) := \chi_{\mathcal{U}}(A) = \begin{cases} 1 & \text{if } A \in \mathcal{U} \\ 0 & \text{otherwise} \end{cases}.$$

Principal ultrafilters correspond to *point masses*, since for some element $x \in X$, $\chi_U(\{x\}) = 1$, i.e. all the mass is concentrated on the point x. Where X is infinite, measures created from free ultrafilters in this way are examples of *diffuse* measures: μ is diffuse if, for all finite sets $F \subseteq X$, $\mu(F) = 0$. (van Douwen, 1992)

Lemma 2.2 If a group G is left amenable, then it is also right amenable, and vice-versa.

Proof Let μ be left invariant, and consider the finitely-additive measure ν defined by $\nu(A) := \mu\left(A^{-1}\right)$. Then ν is right invariant:

$$\nu(Ag) = \mu\left((Ag)^{-1}\right) = \mu\left(g^{-1}A^{-1}\right) = \mu\left(A^{-1}\right) = \nu(A),$$

for any $A \subseteq G, g \in G$, as required. □

Not every left invariant measure is also right invariant, but a left invariant measure and a right invariant measure can be combined to form one which is both left and right invariant (*bi-invariant*).

Lemma 2.3 Suppose G is left and right amenable with finitely-additive measures μ and ν, respectively. There exists a bi-invariant finitely-additive measure ξ.

Proof Define ξ by setting (Wagon, 1993, p148)

$$\xi(A) := \int_{t \in G} \mu\left(At^{-1}\right) d\nu$$

i.e. integrate the map $t \mapsto \mu\left(At^{-1}\right)$ with respect to ν. Showing that ξ is a bi-invariant finitely-additive probability measure on G is omitted here, but the idea is demonstrated in Theorem 4.14. □

Thus, by combining Lemmas 2.2 and 2.3, any group G is seen to be either *amenable* or *non-amenable*.

Example 2.4

1. Every finite group G is amenable. To find a μ, take the *counting measure* given by

$$\mu(E) = \frac{|E|}{|G|},$$

and since the actions of each $g \in G$ on E, being all bijective, cannot change the cardinality, μ satisfies all the required properties.

2. (UL) $(\mathbb{Z}, +) \cong \mathbb{F}_1$ is amenable. Consider setting for all subsets E,

$$\mu(E) = \lim_{n\to\infty}^{??} \frac{|E \cap [-n, n]|}{2n+1}.$$

   This looks promising: all finite subsets therefore have measure 0, but this does not contradict $\mu(\mathbb{Z}) = 1$ because μ is only finitely additive. For infinite E, this expression captures the notion of the amount of space E takes up within $\mathbb{Z}$, which by the limit remains invariant under any (necessarily finite) motion.

   However, there is no guarantee that the limit exists. The sequence is bounded and therefore not divergent, but it is not necessarily convergent. For instance, consider

$$E = \mathbb{Z} \cap (\cdots \cup (-2^{n+1}, -2^n] \cup \cdots (-8, -4] \cup (-2, -1] \\ \cup [1, 2) \cup [4, 8) \cup [16, 32) \cup \cdots \cup [2^n, 2^{n+1}) \cup \cdots).$$

   The sequence with terms $x_n = \frac{|E\cap[-n,n]|}{2n+1}$ in this case does not converge as $n \to \infty$, but does have more than one limit point. It is bounded, so by the Bolzano-Weierstrass theorem it contains at least one convergent subsequence, so one such subsequence can be chosen using an ultralimit: fix a free ultrafilter U on $\mathbb{N}$, and define μ using the ultralimit

$$\mu(E) = U\text{-}\lim_{n\to\infty} \frac{|E \cap [-n, n]|}{2n+1}.$$

   There are infinitely many different free ultrafilters on $\mathbb{N}$, and so there are infinitely many different measures μ that demonstrate the amenability of $\mathbb{Z}$.

3. All compact groups are amenable. This follows from Haar measure theory: if a group is compact, all subsets are compact, and therefore the Haar measure (being an invariant *countably*-additive measure with finite total measure) is defined on all subsets of the group and satisfies the required properties.

4. $\mathbb{F}_2$ is not amenable. Let S_x be the set of words in $\mathbb{F}_2$ with prefix x, and take the generators to be $\{a, b\}$. Assume μ is a measure with the required properties

for amenability. Then since $\mathbb{F}_2 = S_a \uplus aS_{a^{-1}}$ (similarly with b),

$$\begin{aligned}
1 &= \mu(\mathbb{F}_2) \\
&= \mu(\{1\} \uplus S_a \uplus S_b \uplus S_{a^{-1}} \uplus S_{b^{-1}}) \\
&= \mu(\{1\}) + \mu(S_a) + \mu(S_b) + \mu(S_{a^{-1}}) + \mu(S_{b^{-1}}) \\
&= \mu(\{1\}) + \mu(S_a) + \mu(S_b) + \mu(aS_{a^{-1}}) + \mu(bS_{b^{-1}}) \\
&= \mu(\{1\}) + \mu(S_a \uplus aS_{a^{-1}}) + \mu(S_b \uplus bS_{b^{-1}}) \\
&= \mu(\{1\}) + \mu(\mathbb{F}_2) + \mu(\mathbb{F}_2) \\
&\geq 2,
\end{aligned}$$

a contradiction. Hence $\mathbb{F}_2$ cannot be amenable.

5. If a group is amenable, then every subgroup is also amenable. If the subgroup has non-zero measure, this can be demonstrated by scaling the existing measure by a constant. (The zero-measure case is more involved: see Theorem 2.30.) Therefore every group containing a non-amenable group such as $\mathbb{F}_2$ is itself non-amenable.

6. If G is an infinite amenable group with measure μ, then μ is diffuse.

   Proof Suppose there exists some finite set $F \subseteq G$ such that $\mu(F) = k > 0$. G is infinite, therefore there are infinitely many rotated, disjoint copies of F inside G. By invariance, each copy has the same measure k. Finally, by finite-additivity, the disjoint union of any selection of $\lceil 1/k \rceil$ of these copies has measure greater than 1, contradicting $\mu(G) = 1$.

The example of $\mathbb{F}_2$ is used as the foundation for the Banach-Tarski paradox, by translating the self-action into an action on $\mathbb{R}^3$ in a manner that permits emulating the "measure-doubling."

The example of $\mathbb{Z}$ illustrates a few points. Finite additivity, while unwieldy with respect to the study of analysis which prefers countably additive measures, saves the day with respect to finite subsets. The presence of an ultralimit hints at cardinality questions about the set of invariant measures. Finally, the use of an increasing sequence of finite sets (i.e. $[-n, n]$) hints at the underlying reason that all Abelian groups are amenable: every self-action of an Abelian group is like a "shift."

2.2 Invariant means

Various functions can be integrated with respect to various finitely-additive measures. Importantly, invariance of a finitely-additive measure μ can translate into a notion of invariance of the integral operation with respect to μ. Suppose G is an amenable group with finitely-additive measure μ. Since every subset of G is assigned a measure, i.e. $\mathcal{P}(G)$ is measurable by μ, then construction of the Lebesgue integral with respect to μ does not necessarily involve discussion of non-measurable sets or σ-algebras, which simplifies some matters. However, as μ is merely *finitely* additive, certain expected properties of integrals (e.g. the Monotone Convergence Theorem) do not follow (Wagon, 1993, p.147). A widely-cited book that includes a chapter on integrating with respect to measures that are not necessarily countably additive is the popular Dunford and Schwartz, 1958.

Integrals are examples of linear operators. In particular, when taking the integral with respect to some $\mu \in [0,1]^{\mathcal{P}(G)}$, the integral of some bounded function f yields a *mean value* of f across its domain. This leads to the question of what constitutes a "mean value" in a generalised setting. Some properties to consider are as follows. The mean value of a constant function is equal to that constant. If f is non-negative everywhere on its domain, then so too is its mean. Finally, if we sum two functions $f+g$, then the mean of the sum should be the sum of the means, or more accurately, means should be linear.

Day (1957) altered the discourse of amenability by focusing on such a set of linear operators, the *means*, and this is why he later co-opted the word "amenable".[1] A mean m is simply a linear functional with certain properties (described below). Since mean values are an arithmetic idea, we are only really interested in finding the means of bounded real- or complex-valued functions, i.e. those members of the Banach space $\ell^\infty(G)$, and hence $m \in \ell^\infty(G)^*$.[2]

Integrating with respect to a finitely-additive measure provides an example of a mean, and it is possible to go the other way. Each set A corresponds to some $\{0,1\}$-valued function χ_A, which can be given by setting $\chi_A(x) := |A \cap \{x\}|$ for all x. Given a mean m, one may define a finitely-additive probability measure μ by setting $\mu(A) := m(\chi_A)$ for all sets A. The main question is then how invariance

[1] It is customary (Day, 1983; Paterson, 1988, p1; Runde, 2002, p34; Kaimanovich, 2009, p55; Wirzenius, 2012, p1) to mention the term "amenable" was introduced by Day as a pun.

[2] In Day's notation as seen between 1957 and 1968, the mean was written μ, the (semi)group Σ, and the space of bounded functions $m(\Sigma)$.

properties transfer between the two contexts.

This additional machinery may seem like an extra hassle. However, in the context of linear functionals and Banach algebras, we can exploit the fruits of harmonic analysis, and so questions about invariant measures become questions of existence of cluster points and weak convergence.

The following closely mirrors Day (1957), and is expounded upon in detail for the benefit of later observations.

Definition 2.5 (Means) Let G be a group.

1. If $m \in \ell^{\infty}(G)^{*}$ satisfies

$$\inf_{g \in G} \phi(g) \leq m(\phi) \leq \sup_{g \in G} \phi(g) \quad \text{for all } \phi \in \ell^{\infty}(G)$$

   then m is called a *mean*. Note this does not make sense for complex-valued ϕ, and the consideration of complex-valued ϕ is a wrinkle that won't be given much consideration in the present work, but the definition of a mean is adequately generalised by the following. If m satisfies

   (i) $m(\chi_G) = 1$,

   (ii) whenever $\phi(t) \geq 0$ for all $t \in G$, then $m(\phi) \geq 0$ (i.e. m is *non-negative*),

   then m is a mean (Day, 1957; Tao, 2009).

2. If $m \in \ell^{1}(G)$ is non-negative, m is finitely supported, and $\|m\|_1 = 1$, then m is called a *finite mean*.

Remark 2.6 A finite mean m can be viewed as a (non-finite) mean $\hat{m}$ via the canonical embedding of $\ell^{1}(G)$ within $\ell^{\infty}(G)^{*}$, i.e. setting $\hat{m}(\phi) := \langle m, \phi \rangle$ for all $\phi \in \ell^{\infty}(G)$ (Tao, 2009).

The group G could conceivably act on $\ell^{\infty}(G)$ in many ways, but here we care about two actions in particular. The first defines invariance of means, which in turn is used to define amenability. For $g \in G$ and $\phi \in \ell^{\infty}(G)$, let $g \cdot \phi$ and $\phi \cdot g$ denote the left and right actions of g on ϕ respectively, which are given by

$$\{g \cdot \phi\}(t) = \phi(gt), \quad \{\phi \cdot g\}(t) = \phi(tg) \quad \text{for all } t \in G.$$

$g \cdot \phi$	$\phi \cdot g$	Source
$l_g\phi$	$r_g\phi$	Day (1957)
ϕg	$g\phi$	Paterson (1988)
$_{g^{-1}}\phi$	$\phi_{g^{-1}}$	Wagon (1993)

Table 2.1: Other notations for the *natural* or *dual* actions.

In the past, different notation has been used for this action (Table 2.1).

Note in particular that for all $g, t \in G$ and $A \subseteq G$,

$$\left\{g^{-1} \cdot \chi_A\right\}(t) = \chi_A\left(g^{-1}t\right) = \chi_{gA}(t).$$

Definition 2.7 (Amenability–ii) A mean m is said to be *left-invariant* if $m(g \cdot \phi) = m(\phi)$ for all $\phi \in \ell^\infty(G), g \in G$, and likewise, is *right-invariant* if $m(\phi \cdot g) = m(\phi)$.

If a left-invariant mean exists for a given group G, then G is *left-amenable*. Likewise, *right-amenable*, *amenable*, *non-amenable*.

Before demonstrating that the two definitions of amenability described above are equivalent, it is worth mentioning the second interesting action of a (semi)group.

Definition 2.8 On $\ell^1(G)$ we have the *convolution* of $f_1, f_2 \in \ell^1(G)$ given as usually defined:

$$(f_1 * f_2)(x) := \sum_{t \in G} f_1(t) f_2\left(t^{-1}x\right).$$

The convolution operator $*$ induces another action of each $g \in G$ on the left and right of the larger space $\ell^\infty(G)$. For all $g \in G$ and $\phi \in \ell^\infty(G)$, define

$$g * \phi := \chi_{\{g\}} * \phi, \quad \phi * g := \phi * \chi_{\{g\}}.$$

This is the same as merely identifying g with $\chi_{\{g\}}$. Note, however, convolution doesn't work in $\ell^\infty(G)$ in general. For example, if G is an infinite group, $\chi_G * \chi_G$ is infinite everywhere.

For a group G, the two actions above describe the same thing, in the sense that

$$g \cdot \phi = g^{-1} * \phi, \quad \phi \cdot g = \phi * g^{-1} \quad \text{for all } \phi \in \ell^\infty(G), g \in G.$$

In particular, for all $g, t \in G$ and $A \subseteq G$,

$$\{g * \chi_A\}(t) = \left\{g^{-1} \cdot \chi_A\right\}(t) = \chi_A\left(g^{-1}t\right) = \chi_{gA}(t).$$

But $\cdot$ and $*$ are also mutually dual, which is shown as follows. Traditionally, $\phi \in \ell^\infty(G)$ is canonically identified with some $\hat{\phi} \in \ell^1(G)^*$ and *vice-versa*, and consequently $\ell^1(G)^*$ is identified with $\ell^\infty(G)$. This is done by setting

$$\hat{\phi}(f) = \sum_{t \in G} f(t)\,\phi(t) = \langle f, \phi \rangle, \quad \phi(g) = \hat{\phi}\left(\chi_{\{g\}}\right)$$

for all $f \in \ell^1(G)$, $g \in G$. An action on $\ell^\infty(G)$ is dual of some other action on $\ell^1(G)$ if they correspond via the canonical identification.

Lemma 2.9 The left $*$ action on $\ell^1(G)$ and the left $\cdot$ action on $\ell^\infty(G)$ are duals of one another, i.e. for all $g \in G$, $f \in \ell^1(G)$, and $\phi \in \ell^\infty(G)$,

$$\langle \phi, g * f \rangle = \langle g \cdot \phi, f \rangle.$$

Proof

$$\begin{aligned}
\langle \phi, g * f \rangle &= \sum_{t \in G} (g * f)(t)\,\phi(t) \\
&= \sum_{t \in G} f\left(g^{-1}t\right)\phi(t) \\
&= \sum_{t \in G} f(t)\,\phi(gt) \\
&= \sum_{t \in G} f(t)\,(g \cdot \phi)(t) \\
&= \langle g \cdot \phi, f \rangle,
\end{aligned}$$

as required. □

Similarly, if we try to apply the $*$ action on $\ell^\infty(G)$, the dual in $\ell^1(G)$ is again the $\cdot$ action.

Since the convolution action $*$ has a convenient name ("convolution"), the other action $\cdot$ is sometimes called the *dual* action, or the *natural* action.

Importantly, Lemma 2.9 strongly links amenability from the perspective of $\ell^\infty(G)$ with amenability from the perspective of $\ell^1(G)$, and $\ell^1(G)$ is a Banach algebra when one includes convolution. Perhaps amenability can be thought of as solely a Banach algebra property? More on that later.

The following would be expected by the use of the same term for two different conditions.

Theorem 2.10 The two definitions of amenability above (Definitions 2.1 and 2.7) are equivalent.

Proof Let G be a group. Suppose that G supports an invariant finitely-additive measure μ. Let m be defined by setting

$$m(f) := \int_G f d\mu$$

for all $f \in \ell^\infty(G)$. It is straightforward to verify that m inherits the properties listed in Definition 2.7:

1. m is a linear functional.

   PROOF: For $f_1, f_2 \in \ell^\infty(G)$ and $\lambda \in \mathbb{C}$,
   $$\begin{aligned} m(f_1 + \lambda f_2) &= \int_G (f_1 + \lambda f_2) d\mu \\ &= \int_G f_1 d\mu + \lambda \int f_2 d\mu \\ &= m(f_1) + \lambda m(f_2). \end{aligned}$$
2. $m(\chi_G) = \int_G 1 d\mu = 1$.
3. m is non-negative.

   PROOF: For a non-negative $f \in \ell^\infty(G)$, the integral is a supremum of sums of non-negative simple functions only, and is therefore non-negative.
4. m is invariant.

   PROOF: This will not be shown in full, as similar working will be used in detail in later chapters. The key idea is that for every set $A \subseteq G$,
   $$m(\chi_A) = \int_G \chi_A d\mu = \mu(A)$$
   by definition, and therefore
   $$\begin{aligned} m\left(g^{-1} \cdot \chi_A\right) &= m(\chi_{gA}) \\ &= \mu(gA) \\ &= \mu(A) \\ &= m(\chi_A). \end{aligned}$$

It is then a matter of verifying m is invariant at each step of the integral construction up to arbitrary functions $f \in \ell^\infty(G)$ (and it is).

Conversely, suppose G supports an invariant mean m. For all sets $A \subseteq G$, define $\mu(A) := m(\chi_A)$. It is straightforward to verify the properties listed in Definiton 2.1 as follows:

1. $\mu(G) = m(\chi_G) = 1$.
2. μ is finitely additive.

   PROOF: For any disjoint sets $A, B \subseteq G$:

$$\begin{aligned}\mu(A \uplus B) &= m\left(\chi_{(A \uplus B)}\right)\\ &= m(\chi_A + \chi_B)\\ &= m(\chi_A) + m(\chi_B) \quad \because \text{linearity of } m\\ &= \mu(A) + \mu(B),\end{aligned}$$

   as required.
3. μ is invariant.

   PROOF: For any $g \in G, A \subseteq G$:

$$\begin{aligned}\mu(gA) &= m(\chi_{gA})\\ &= m\left(g^{-1} \cdot \chi_A\right)\\ &= m(\chi_A) \quad \because m \text{ is invariant}\\ &= \mu(A),\end{aligned}$$

   as required. □

Note that $m(A)$ is often used as shorthand for $m(\chi_A)$.

2.3 Almost-invariant finite means

Most other flavours of amenability are based upon the following. An argument of Namioka (1964), itself a refinement of an argument going back to Følner (1955), relates the infinite means to *almost*-invariance: collections of finite means which become invariant in some kind of limit. This was helpfully summarised in notes by Tao (2009) and his proof is fleshed out below, with a minor simplification.

Theorem 2.11 If G is an amenable group, then for every $x \in G$ and $\epsilon > 0$ there exists a finite mean ν such that

$$\|\nu - x * \nu\|_1 < \epsilon.$$

Proof This is shown by contradiction. Let $\Phi(G)$ denote the space of finite means. Assume that G is amenable with a mean $m \in \ell^\infty(G)^*$, and that there exists an $x \in G$ and $\epsilon > 0$ such that for every finite mean $\nu \in \Phi(G)$, $\|\nu - x * \nu\|_1 \geq \epsilon$.

1. The set $\{(\nu - x * \nu) : \nu \in \Phi(G)\}$ is convex in $\ell^1(G)$ and bounded away from $\{0\}$.
   PROOF: Φ is convex, and this translates to the elements of the form $\nu - x * \nu$. The set is bounded away from 0 by hypothesis.
2. There exists a functional $\rho \in \ell^1(G)^*$ such that $\rho(\nu - x * \nu) \geq 1$ for all finite means ν.
   PROOF: The previous step demonstrates the two disjoint convex subsets of $\ell^1(G)$ required for applying the Hahn-Banach separation theorem.
3. There exists a function $r \in \ell^\infty(G)$ such that $\langle \nu, (r - x \cdot r)\rangle \geq 1$ for all finite means ν.
   PROOF: It follows from the established identification $\ell^1(G)^* \equiv \ell^\infty(G)$ that there is an $r \in \ell^\infty(G)$ satisfying $\rho(\nu) = \langle r, \nu\rangle$, and then
$$\begin{aligned} 1 &\leq \rho\left(\nu - x * \nu\right) \\ &= \langle r, \nu - x * \nu\rangle \\ &= \langle r, \nu\rangle - \langle r, x * \nu\rangle \\ &= \langle r, \nu\rangle - \langle x \cdot r, \nu\rangle \\ &= \langle r - x \cdot r, \nu\rangle . \end{aligned}$$
4. $(r - x \cdot r) \geq 1$ pointwise.
   PROOF: Use $\nu = \chi_{\{t\}}$ for each $t \in G$: then $\left\langle r - x \cdot r, \chi_{\{t\}}\right\rangle = (r - x \cdot r)(t)$.
5. The mean m is not left invariant, contradicting the assumption that G is amenable.
   PROOF: Applying m to the previous step, we get
$$m(r) - m(x \cdot r) = m(r - x \cdot r) \geq m(1) = 1,$$
   which shows that m is not invariant under the action of x on r. □

Remark 2.12 While the proof ends by specialising all finite means to χ_t for all $t \in S$, one necessarily involves finite means at the beginning since $\Phi(S)$ is the convex hull of $\{\chi_t : t \in S\}$, which is not, itself, convex.

The above theorem can be strengthened slightly by introducing a finite set K. This straightforward improvement gives: if G is amenable then for every finite $K \subseteq G$ and $\epsilon > 0$ there is a finite mean ν with $\|\nu - x * \nu\|_1 < \epsilon$ for every $x \in K$ (Namioka, 1964). The proof above only needs the appropriate spaces to be augmented by K, i.e. the element $(\nu - x * \nu) \in \ell^1(G)$ becomes a bundle of such elements: $(x \mapsto$

$(\nu - x * \nu)) \in \ell^1(G)^S$.

Such a strengthened condition is more useful for demonstrating that a group is *not* amenable than it is in showing a group *is* amenable.

2.4 Følner criteria

Using a "layer-cake decomposition" of the finite means, Theorem 2.11 above can be reduced to a statement about finite sets, known as a *Følner criterion*.

Theorem 2.13 If G is an amenable group, then for all $x \in G$ and $\epsilon > 0$ there exists a finite set F such that

$$\frac{|xF \triangle F|}{|F|} \leq \epsilon.$$

Proof Brief sketch: By taking the layer-cake decomposition of the required finite mean ν, we can see at least one of the layer sets E_k is a finite set satisfying the Følner condition.

Fix an $x \in G$ and an $\epsilon > 0$.

1. By Theorem 2.11, there exists a finite mean ν such that
$$\|\nu - x * \nu\|_1 \leq \epsilon.$$
2. Since $\mathrm{supp}(\nu)$ is finite and ν is bounded, we may fix a layer-cake decomposition of ν, i.e. fix n nested finite sets $E_1 \subset E_2 \subset \cdots \subset E_n$ and positive finite numbers $c_1, \ldots, c_n$ such that
$$\nu = \sum_{i=1}^{n} c_i \chi_{E_i}.$$
In this case, $x * \nu$ will have the layer-cake decomposition $x * \nu = \sum_{i=1}^{n} c_i \chi_{xE_i}$.
3. $\sum_{i=1}^{n} c_i |E_i| = 1$.
Proof: By definition $\|\nu\|_1 = 1$, so
$$1 = \|\nu\|_1 = \sum_{t \in G} \nu(t) = \sum_{i=1}^{n} c_i |E_i|.$$
4. For each $k \in \{1, \ldots, n\}$ and $x \in S$, $|\nu(t) - (x * \nu)(t)| \geq c_k$ for all $t \in xE_k \triangle E_k$.
Proof: Since the E_i are nested, if $t \notin E_k$ then $\nu(t) \leq \sum_{i=1}^{k-1} c_i$. Then either $t \in E_i \backslash xE_i$ and so $\nu(t) = \sum_{i=1}^{k} c_i$ and $(x * \nu)(t) \leq \sum_{i=1}^{k-1} c_i$, or $t \in xE_i \backslash E$ and so $\nu(t) \leq \sum_{i=1}^{k-1} c_i$ and $(x * \nu)(t) = \sum_{i=1}^{k} c_i$.
5. Using steps 3 and 4,
$$\sum_{i=1}^{n} c_i |xE_i \triangle E_i| \leq \|\nu - x * \nu\|_1 \leq \epsilon = \epsilon \cdot \sum_{i=1}^{n} c_i |E_i|.$$

6. Thus for at least one $k \in \{1, \ldots, n\}$,
$$|xE_k \triangle E_k| \leq \epsilon |E_k|,$$
so the finite set E_k is the required set. □

Remark 2.14 Similarly to Theorem 2.11, Theorem 2.13 can be slightly improved: if a group G is amenable, then for all finite sets $K \subseteq G$ and $\epsilon > 0$ there exists a finite set F such that $|xF \triangle F| / |F| \leq \epsilon$ for all $x \in K$.

Note that since each $x \in G$ acts in a bijective manner on G,

$$2|F \backslash xF| = |F \triangle xF| = 2|xF \backslash F|$$

for any finite set $F \subseteq G$, and therefore $|xF \triangle F|$ can be replaced with either $|F \backslash xF|$ or $|xF \backslash F|$.

To close the loop with invariant measures and means, we need to show that if a group satisfies the Følner condition, then it is amenable, which Lemma 2.15 accomplishes.

Lemma 2.15 For any countable group G, G is amenable if and only if there exists a sequence $\{F_n\}_{n \in \mathbb{N}}$ (a *Følner sequence*) of finite subsets of G that "eventually cover G", i.e. for every $h \in G$, h is also in F_j for all $j > i$ for some i, and that satisfy

$$\lim_{n \to \infty} \frac{|F_n \triangle gF_n|}{|F_n|} = 0 \quad \text{for each } g \in G.$$

Proof The proof that an amenable group always has a Følner sequence is an easy consequence of Theorem 2.13: take $\epsilon = \frac{1}{n}$. This provides the n-th finite set of the sequence, F_n.

Conversely, intuitively we want to use a sequence with the term

$$x_n(A) = \frac{|A \cap F_n|}{|F_n|}$$

in a limit, but as demonstrated earlier, the limit may not exist. Again, fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ containing the intervals $[n, \infty)$ for each $n \in \mathbb{N}$. Then set up the finitely-additive measure μ by using the ultralimit

$$\mu(A) := \lim_{\mathcal{U}} x_n(A) = \lim_{\mathcal{U}} \frac{|A \cap F_n|}{|F_n|}.$$

1. The ultralimit, and hence μ, always exists.

   PROOF: The sequence is bounded, so by the Bolzano-Weierstrass Theorem there exists a convergent subsequence. See Chapter 1.

2. μ is a finitely-additive probability measure.

   2.1. $\mu(S) = 1$

   PROOF: $x_n(S) = \frac{|F_n|}{|F_n|} = 1$ for all n.

   2.2. $\mu(A \uplus B) = \mu(A) + \mu(B)$ for all disjoint $A, B \subseteq S$.

   PROOF: Suppose $A \cap B = \emptyset$. Then

$$\begin{aligned}\mu(A \cup B) &= \lim_{\mathcal{U}} \frac{|(A \cup B) \cap F_n|}{|F_n|} \\ &= \lim_{\mathcal{U}} \frac{|A \cap F_n| + |B \cap F_n|}{|F_n|} \\ &= \lim_{\mathcal{U}} \frac{|A \cap F_n|}{|F_n|} + \lim_{\mathcal{U}} \frac{|B \cap F_n|}{|F_n|} \qquad \because \text{Lemma 1.11} \\ &= \mu(A) + \mu(B).\end{aligned}$$

3. μ is invariant.

   PROOF: :et $\{F_n\}_{n\in\mathbb{N}}$ be the Følner sequence. Let $x'_n = |gA \cap F_n|/|F_n|$. Note that $|A \cap B| = |g(A \cap B)| = |gA \cap gB|$ for any sets $A, B \subseteq G$ and $g \in G$. Then,

$$\begin{aligned}|x'_n - x_n| &= \left|\frac{|gA \cap F_n|}{|F_n|} - \frac{|A \cap F_n|}{|F_n|}\right| \\ &= \left|\frac{|gA \cap F_n|}{|F_n|} - \frac{|gA \cap gF_n|}{|F_n|}\right| \\ &= \frac{||gA \cap F_n| - |gA \cap gF_n||}{|F_n|} \\ &\leq \frac{|gA \cap (F_n \triangle gF_n)|}{|F_n|} \\ &\leq \frac{|F_n \triangle gF_n|}{|F_n|} \to 0 \text{ as } n \to \infty\end{aligned}$$

   by hypothesis (Følner criterion), therefore $\mu(A) = \mu(gA)$ for any g, A. □

2.4.1 Variations on Følner criteria

An alternative Følner criterion, and in fact one of the most popular in modern literature, is described in the next theorem.

Theorem 2.16 (Følner Criterion) A group G is amenable iff for every $\epsilon > 0$ and finite subsets $K, S \subseteq G$ there exists a finite subset F such that $S \subseteq F$ and $|FK \backslash F| < \epsilon |F|$. Such an F is called a *Følner set* (Akhmedov, 2009). (Not shown here.)

Picking apart the components of this theorem:

- The set S ensures F is arbitrary large and eventually covers all of G.
- The set K rotates the a copy of the set F, one for each $k \in K$.
- The quantity $|FK \backslash F|$ therefore measures how much more of G can be covered by rotating F about in finitely many directions. When the elements in K are short (say, a set of generators), this is approximately the size of the boundary of F as F becomes large.
- It follows that the ratio $|FK \backslash F| \,/\, |F|$ is approximately the ratio between the surface area and the volume of F, as F becomes large.

An advantage to the Følner Criterion is that it is relatively easier to show a group is amenable than with the primal definition, since one simply shows the existence of a Følner *sequence* (of Følner sets). However, it is often impractical to use it for showing a group is *not* amenable. As asserted by Akhmedov (2009), one would have to argue that given a small ϵ there are no ϵ-Følner sets (see Examples 2.19 and 2.20).

Følner's original results are stated below, with minor rewording from the original paper (Følner, 1955).

Theorem 2.17 (Følner's necessary condition) A necessary condition that a group G is amenable is that for every k in the interval $0 < k < 1$, and arbitrary, finitely many, elements $a_1, \ldots, a_n$ from G, there exists a finite subset E of G such that

$$|E \cap Ea_i| \geq k\,|E| \quad \text{for} \quad i = 1, \ldots, n.$$

(Not shown here)

Theorem 2.18 (Følner's sufficient condition) A sufficient condition that a group G is amenable is that there exists a number k_0 in the interval $0 < k_0 < 1$ such that for arbitrary, finitely many, not necessarily different, elements $a_1, \ldots, a_n$ from G there exists a finite subset E of G such that

$$\frac{1}{n}\sum_{i=1}^{n} |E \cap Ea_i| \geq k_0\,|E|\,.$$

(Not shown here)

Condition	Source
$\forall \epsilon, a_1 \ldots a_n \exists F : \lvert F \cap Fa_i \rvert \geq \epsilon \lvert F \rvert$	Følner (1955)
$\exists \epsilon_0 \forall a_1 \ldots a_n \exists F : n^{-1} \sum_{i=1}^{n} \lvert F \cap Fa_i \rvert \geq \epsilon_0 \lvert F \rvert$	Følner (1955)
$\lim_{i\to\infty} \lvert gF_i \triangle F_i \rvert / \lvert F_i \rvert = 0$	
$\forall \epsilon, K, S \exists F : S \subseteq F, \lvert FK \backslash F \rvert / \lvert F \rvert < \epsilon$	Akhmedov (2009)

Table 2.2: Summary of the variations on Følner criteria for discrete groups. $\epsilon, \epsilon_0 > 0$, the $a_1 \ldots a_n$, g are elements of G, and F, F_i, K, S are finite subsets of G.

Følner remarks that, for groups, each condition implies the other.[3]

Example 2.19 $\mathbb{Z}$ is amenable, but this time we're going to prove it with Følner sets. In particular, we'll use the sequence $\{F_n\}_{n\in\mathbb{N}}$ where $F_n = [-n, n] \cap \mathbb{Z}$. This will suffice as a Følner sequence for all elements $k \in \mathbb{Z}$. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{\lvert (k + F_n) \triangle F_n \rvert}{\lvert F_n \rvert} &= \lim_{n\to\infty} \frac{\lvert [-n + k, n + k] \triangle [-n, n] \rvert}{2n + 1} \\ &= \lim_{n\to\infty} \frac{2k}{2n + 1} \\ &= 0.\end{aligned}$$

Example 2.20 $\mathbb{F}_2$ we know is not amenable, nevertheless it is worth using it to demonstrate a Følner criterion. To show this group is non-amenable this way is not intuitive, and here Lemma 2.15 is repeated by stealth.

Suppose $\mathbb{F}_2$ is amenable and has the generating set $\{a, b\}$. Therefore it satisfies the Følner condition: for each $\epsilon > 0$ and finite set $K \subseteq \mathbb{F}$, there is a finite F satisfying $\lvert xF \triangle F \rvert < \epsilon \lvert F \rvert$ for all $x \in K$. Let $K = \left\{a, b, a^{-1}, b^{-1}\right\}$ and recall the sets S_a, S_b,

[3] Percolation theory also begain in the 1950s. The concept of *density* is very similar in nature to the kind of measure obtained via the Følner condition: a subset S of the integer lattice $\mathbb{Z}^d$ has density α when for every increasing sequence of finite rectangles $\{R_n\}$ eventually covering all of $\mathbb{Z}^d$, the following limit exists (Burton and Keane, 1989):

$$\alpha = \lim_{n\to\infty} \frac{\lvert S \cap R_n \rvert}{\lvert R_n \rvert}.$$

It is again possible that this limit does not exist.

$S_{a^{-1}}$, and $S_{b^{-1}}$ given as previously. Note that for any finite set F,

$$\begin{aligned} a(F \backslash S_{a^{-1}}) &= a(F \cap (\{1\} \cup S_a \cup S_b \cup S_{b^{-1}})) \\ &= aF \cap S_a \end{aligned}$$

thus for all $\epsilon > 0$ there exists a finite set F such that

$$\begin{aligned} |F| - |F \cap S_{a^{-1}}| &= |F \backslash S_{a^{-1}}| \\ &= |a(F \backslash S_{a^{-1}})| \\ &= |aF \cap S_a| \\ &\leq ||aF \cap S_a| - |F \cap S_a|| + |F \cap S_a| \\ &= |(aF \triangle F) \cap S_a| + |F \cap S_a| \\ &\leq |aF \triangle F| + |F \cap S_a| \\ &\leq \epsilon\,|F| + |F \cap S_a| \quad \because \text{Følner cond. by assumption.} \end{aligned}$$

This can be repeated for a^{-1}, b, and b^{-1}. Summing together each inequality gives

$$\begin{aligned} 4\,|F| - \sum_{x \in K} |F \cap S_{x^{-1}}| &\leq 4\epsilon\,|F| + \sum_{x \in K} |F \cap S_x| \\ \therefore \quad 4\,|F| - |F| &\leq 4\epsilon\,|F| + |F| \\ \therefore \quad \frac{1}{2} &\leq \epsilon, \end{aligned}$$

contradicting $\epsilon > 0$ being arbitrary. (This proof adapted from Tao (2009).) □

Remark 2.21 For a group G let a measure μ or mean m be called *density-like* if it is expressible as a limit such as the one in Lemma 2.15: that is, there are increasing sequences of finite sets, and to measure a set A or bounded function ϕ one considers a limit of a ratio of cardinatilies or norms. Prior to these results we may have suspected that there may have been some amenable group G in which every invariant measure or mean was not density-like. An interpretation of the Følner Condition is that every group that is amenable must be amenable via some density-like invariant measure/mean. Unfortunately, while density-like measures and means are easy to conceptualise, the proof via the Hahn-Banach separation theorem given above is not constructive.

2.5 Amenability and growth

The concept of growth in relation to groups is said to have first appeared in the mid 1950s (Grigorchuk, 1991), and as it turned out, growth is highly relevant to amenability theory.

The Følner criterion indicates that amenability is intimately related to growth in a group. As remarked above, within the Cayley graph of the group, the increasing finite sets F have “volume” $|F|$ and “surface area” $|\partial F| \approx |FK \backslash F|$. For an increasing sequence of compact solids in some space, intuitively, the rate of change of the surface area is proportional to the derivative of the rate of change of the volume. An amenable group necessarily has that ratio converging to zero for all Følner sequences, but there may be non-Følner sequences where this is not the case. The Følner condition therefore provides sequences that describe a lower bound for the growth of the group. For any non-amenable group there is some sequence for which the ratio does not converge to zero, indicating a function for the volume proportional to its own derivative: an exponential.

Often, the sequence of open balls of radius n in the Cayley graph of the group functions as a Følner sequence. The growth rate of a group is based on this sequence.

Definition 2.22 (Growth rate for groups) Let $w \in \mathbb{F}_S$. Each w is the product of some finite number of generators from the symmetrised generating set $S \cup S^{-1}$:

$$w := \prod_{i=0}^{n} s_{k_i}^{m_i}.$$

Define $|w|$ to be *word length of* w,

$$|w| = \sum_{i=0}^{n} |m_i| .$$

If 1 is the identity of $\mathbb{F}_S$, then $|1| = 0$. Each w corresponds to a vertex in the Cayley graph $\Gamma(\mathbb{F}_S)$, or alternatively, a unique path starting at the origin/1 vertex. It is easy to use the word length to define a metric on the Cayley graph.

Let $G = \langle S | R \rangle$ be finitely-presented, i.e. S, R are finite and

$$1 \to \langle R \rangle^{\mathbb{F}_S} \to \mathbb{F}_S \xrightarrow{\phi} G \to 1$$

is exact[4]. ϕ is surjective, so for any $x \in G$, there is at least one (and usually more than one) $w \in \mathbb{F}_S$ such that $w\phi = x$.

We are concerned with the number of elements of G with smallest word with length n. In case an element of G has multiple words of length n, we need to pick only one. In other words, if $[x] = \phi^{-1}(\{x\})$ denotes the equivalence class of words in $\mathbb{F}_S$ equal to x under the quotient, then for all $x \in G$, $|x| := \min_{w\in[x]} |w|$.

Let $B_n(G, S) = \{x \in G : |x| \leq n\}$, i.e. the *filled ball of radius* n *centred at the origin* in the group G relative to the generating set S. (G and S determine R.) For simplicity let $B(n)$ denote $B_n(G, S)$ where G, S are determined from the context. The *growth function of* G *relative to* S is

$$g(n) := |B_n(G, S)| = |B(n)|$$

Clearly every $g(n)$ is monotone non-decreasing.

Let $\sim$ be the equivalence relation on functions $a, b : \mathbb{R} \to \mathbb{R}$ where $a \sim b$ if there exists a finite constant C such that $a(n/C) \leq b(n) \leq a(Cn)$ as n becomes large. $\sim$ captures the notion that constant factor differences in growth are insubstantial, similar to big-Oh notation in complexity.

The *growth class of* G is one of *constant/polynomial/exponential* if the equivalence class of functions containing the function $g : n \mapsto |B(n)|$ also contains a constant/polynomial/exponential function. (The classes are disjoint!) The growth class of G is *intermediate* if it exceeds any polynomial, but is exceeded by any exponential (as n becomes large). A theorem (not proven here) is that the growth class for G does not depend on the choice of generators S.

Alternatively we could formulate the above in terms of

$$S_n(G, S) = \{x \in G : |x| = n\} = B(n) \setminus B(n-1).$$

Some examples for different growth classes are in Table 2.3.

The constant, polynomial, and intermediate classes are grouped together as *subexponential*, including all functions exceeded by the exponential class for large n. This grouping is natural considering this next result.

Theorem (AC) 2.23 Finitely-generated groups which have subexponential growth

[4]Recall that a sequence of group homomorphisms is *exact* if the image of each arrow is the kernel of the next arrow.

Growth class	Examples	Source
Constant	Finite groups	
Polynomial	$\mathbb{Z}^d$, H_3, ...	Grigorchuk (1991)
Intermediate	Grigorchuk's examples	Grigorchuk (1991), Grigorchuk (1996)
Exponential	$\mathbb{F}_2$, F, $\mathbb{Z}_2 \wr \mathbb{Z}$,...	

Table 2.3: Examples of groups in different growth classes

are supramenable (Wagon, 1993, p.192), and are therefore amenable. (Not shown here.)

Thus we focus on groups of exponential growth. If G has exponential growth, then the quantity

$$g_G = \lim_{n\to\infty} g(n)^{1/n}$$

is the *growth rate* of G, and obviously G has exponential growth if $g_G > 0$.

There is related quantity that aids in refining the exponential growth class, with an eye to amenability in particular. The following is from Burillo, Cleary, and Wiest (2007).

Definition 2.24 (Cogrowth) Let $1 \to K \to \mathbb{F}_m \to G \to 1$ be a presentation of G (so G has m generators). The *cogrowth* of G is the growth of the normal subgroup K of $\mathbb{F}_m$, as measured within $\mathbb{F}_m$.

The *cogrowth function* of G is

$$\bar{g}(n) = |B(n) \cap K|$$

where $B(n)$ is the filled ball of radius n in $\mathbb{F}_m$, and since in many cases the cogrowth function will be exponential, we define the *cogrowth rate* of G, γ_G, to be given by

$$\gamma_G := \limsup_{n\to\infty} \bar{g}(n)^{1/n}.$$

Obviously, $1 \le \gamma_G \le 2m - 1$.

Example 2.25 1. Consider $\mathbb{Z}_3$, which has the presentation $1 \to 3\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_3 \to 1$. The cogrowth function is then $\bar{g}(n) = |[-n, n] \cap 3\mathbb{Z}|$, and therefore $\bar{g}(n) = 2\left\lfloor \frac{n}{3} \right\rfloor + 1$. Being only linear, $\gamma = 1$.

2. Consider $\mathbb{Z} \times \mathbb{Z}$. This group has the presentation $1 \longrightarrow [\mathbb{F}_2, \mathbb{F}_2] \longrightarrow \mathbb{F}_2 \longrightarrow \mathbb{Z} \times \mathbb{Z} \longrightarrow 1$ (the Abelianisation of $\mathbb{F}_2$ is $\mathbb{Z} \times \mathbb{Z}$). $[\mathbb{F}_2, \mathbb{F}_2]$ is an infinitely-generated subgroup of infinite index in $\mathbb{F}_2$. For $\mathbb{F}_{\{a,b\}}$ it is freely generated by the set $\{[a^m, b^n] : m, n \in \mathbb{Z} \setminus \{0\}\}$. It turns out that $\gamma = 3$ as we'll see shortly.

The following cogrowth characterisation of amenability is taken from Burillo, Cleary, and Wiest (2007), and was originally devised by Kesten (1959), who was at the time recently concerned with symmetric random walks on groups (Kesten, 1958). Kesten's proof used Følner's now-famous criterion, but was at the time only recently-proven.

Theorem 2.26 (Kesten's cogrowth criterion) Let G be a finitely-generated group with finite set of generators S, and cogrowth rate γ_G. G is amenable if, and only if, $\gamma_G = 2|S| - 1$. (Not proven here.)

Example 2.27

1. The free groups themselves are easy examples, since the group generated by the (empty) set of relators is trivial. For the free group $\mathbb{F}_n$, $g(n) = 1$ for all n, so $\gamma = 1$. Thus $\mathbb{F}_n$ is amenable if and only if $1 = 2|S| - 1 = 2n - 1$ i.e. $n = 1$.

2. Consider the finite group $\langle a, b \,|\, a^2, b^2, [a, b] \rangle \cong \mathbb{Z}_2^2$. It is finite, therefore it is amenable, therefore its cogrowth $\gamma = 3$. On the other hand, it is not as easy to demonstrate that $\gamma = 3$ by manually counting elements in the free subgroup generated by the relators.

This has an interpretation in terms of symmetric random walks on the group. Equivalently, think of this geometrically as walks on the Cayley graph. Loosely speaking, a group is amenable if for any random walk of length L starting at 1, the probability of returning to the identity decreases slower than exponentially with L.

Question 2.28 Are questions about growth, cogrowth, and random walks and so on interesting for non-groups?

Growth and cogrowth are innately related to the Cayley graph of the group—balls in the "word space" corresponding to balls in the graph—and the Cayley graph is necessarily the 1-skeleton of the immediate 2-complex having fundamental group isomorphic to the group in question. This places amenability as a geometric concern in group theory, via growth.

Exponential growth is related to hyperbolic groups, in particular in relation to *growth tightness*. Hyperbolic groups are explained briefly as follows (Gromov, 1987). A geodesic triangle is δ-*slim* if each side is contained in a δ-neighborhood of the other two sides. A δ-*hyperbolic group* is then one where all geodesic triangles in the Cayley graph are δ-slim, and a hyperbolic group is simply a δ-hyperbolic for some $\delta > 0$. Hyperbolicity of a group is thus similar to the geometric conception of hyperbolicity (negative curvature).

Trivially, all finite groups are hyperbolic. Cyclic and virtually cyclic groups (groups with a finite-index[5] cyclic subgroup) are also hyperbolic. Every finitely-generated free group is hyperbolic.

A non-hyperbolic group is $\mathbb{Z} \times \mathbb{Z}$, which is generalised to the Baumslag-Solitar groups, denoted $B(m, n)$, that are also not hyperbolic ($\mathbb{Z} \times \mathbb{Z} \cong B(1, 1)$). But similar to the case with $\mathbb{F}_2$ and amenability, any finitely-generated group containing an isomorphic copy of $B(m, n)$ for any m, n is not hyperbolic.

Remark 2.29 Growth and cogrowth of a group are only partially related. Consider $\mathbb{Z}^2 = \langle a, b | [a, b] \rangle$. The number of elements of length n, $\gamma(n) = 4n$, reflects the fact that the cogrowth is neither trivial nor maximal, but the cogrowth is not simply the growth of $\mathbb{F}_2$, $4 \cdot 3^{n-1}$, minus the growth of $\mathbb{Z}^2$. Nontrivial elements of $\langle [a, b] \rangle^{\mathbb{F}_2}$ have length at least 4, whereas the relator $[a, b]$ impacts the growth as early as the words with length 2.

2.6 Classical amenability results for groups

There are a number of easy results that, collectively, show that non-amenability is a somewhat rare thing.

Theorem 2.30 (Classical results for groups) The following are true for countable groups.

(i) (AC) Subgroups and quotients of amenable groups are amenable.

(ii) (AC) The union of a directed system of amenable groups is amenable.

(iii) (AC) A group is *locally amenable* when every finitely-generated subgroup is amenable. Local amenability implies amenability. In particular, locally finite groups are amenable.

[5] Recall that the *index of* H *in* G, denoted $|G : H|$, is the number of cosets of H.

(iv) (AC) Every Abelian group is amenable.

(v) If $1 \to G/H \to G \to H \to 1$ is exact and both H and G/H are amenable, then G is amenable.

(vi) The direct product of finitely many amenable groups is amenable.

(vii) A group is *virtually amenable* if it has a finite index amenable subgroup. Virtual amenability implies amenability.

(viii) Solvable groups are amenable, and by the previous point, virtually solvable groups are ameanble. Examples of solvable groups include nilpotent groups.

(ix) Countable discrete groups that contain $\mathbb{F}_2$ are *not* amenable.

Proof sketches

(i) (Wagon, 1993, p149) For subgroups $H \subseteq G$, if $\mu(H) > 0$ this is trivial, otherwise we can rely on choosing a representative from each of the cosets of the subgroup as follows. If H is the subgroup of G, and μ is the given invariant finitely-additive measure on G, choose a set M of representatives of each coset of H and then let ν be the finitely-additive measure on H given by

$$\nu(A) := \mu(MA).$$

An alternative jusitification: amenability is equivalent to the Følner condition, and since the same Følner sequences must eventually cover the subgroup, it follows that the subgroup has the Følner property and is therefore amenable.

For the quotient G/H, observe that $\nu : G/H \to [0, 1]$ given by setting

$$\nu(A) = \mu\left(\bigcup A\right)$$

suffices.

(ii) For each group G_i in the directed system, the subset of the compact space $[0, 1]^{\mathcal{P}(G)}$ (compact due to Tychonoff's Theorem) consisting of invariant finitely-additive probability measures restricted to G_i is closed. The directed set of such subsets has the finite intersection property, and therefore the intersection of all such sets is nonempty. (Compare with Theorem 5.47.)

(iii) Any group is the direct union of its finitely-generated subgroups, therefore the previous point applies.

(iv) An Abelian group is the direct union of finitely-generated Abelian groups, and such groups are direct products of primary and infinite cyclic groups—this is the Fundamental Theorem of Finitely-Generated Abelian Groups. Since primary and infinite cyclic groups are easily seen to be amenable, every finitely-generated Abelian group is amenable, and in turn every Abelian group is amenable. An alternative proof (that generalises to semigroups) is given in Theorem 4.17.

(v) This involves being clever about combining the invariant means on H and G/H, with a similar idea to part (i) above. See Ceccherini-Silberstein and Coornaert (2010, p90), and compare with Theorem 5.25.

(vi) For the product $G = G_1 \times G_2$, let $H = \{(g_1, 1_{G_2}) : g_1 \in G_1\}$. H is a normal subgroup of G and $G/H \cong G_2$. The previous point then applies.

(vii) If H is the finite-index subgroup of G, and H has the finitely-additive invariant measure ν, then define μ on G by

$$\mu(A) := \nu(A \cap H) .$$

(viii) By definition the derived series of a solvable group is finite and terminates in the trivial group. An inductive argument works by starting at the trivial group and extending up via part (v) above to get to the original group.

(ix) This is a trivial consequence of part (i). □

Some good notes and proofs on some of the above results were posted by Tao (2009).

Remark 2.31 The *elementary amenable* groups are groups that can be composed from finite and Abelian groups by a sequence of operations that maintain amenability, namely, taking subgroups, quotients, extensions, and directed unions. It follows from the above theorem that all elementary amenable groups are, indeed, amenable.

- Let NF denote the class of groups not containing a subgroup isomorphic to $\mathbb{F}_2$.

- Let AG denote the class of all amenable groups.

- Let EG denote the class of elementary amenable groups. By definition EG is the smallest class closed under the operations contining the finite and Abelian groups.

It is obvious that

$$\mathsf{EG} \subseteq \mathsf{AG} \subseteq \mathsf{NF},$$

but that both inclusions are strict eluded proof for some time.

Day conjectured that $\mathsf{EG} = \mathsf{AG}$, which was reasonable since at that time all known amenable groups were elementary. Grigorchuk demonstrated one first counterexample to $\mathsf{EG} = \mathsf{AG}$ of a finitely-presented amenable group that is not in EG in 1983 (Grigorchuk, 1996). This example was not finitely presented, and at that point, $\mathsf{EG} = \mathsf{AG}$ held for all finitely presented groups, so a subsequent conjecture was that $\mathsf{EG} = \mathsf{AG}$ held for the finitely presented groups. A second counterexample in Grigorchuk (1996) demonstrated a counterexample to this conjecture as well.

The *von Neumann conjecture* is that $\mathsf{AG} = \mathsf{NF}$, and is also false in general, though counterexamples are again somewhat rare. The known counterexamples include certain free Burnside groups (being periodic, they do not contain $\mathbb{F}_2$), the Tarski monster groups created by Ol'shanskii in 1980, and the finitely-presented group proven in a 110-page marathon paper by Ol'shanskii and Sapir (2002).

Prior to these counterexamples, Bass and Serre conjectured that, over a characteristic-0 field, a linear group is either in NF or it is virtually solvable. The *Tits alternative* is a theorem of Tits (1972) showing that precisely this conjecture is true, and therefore that $\mathsf{NF} = \mathsf{AG}$ within the class of linear groups.

2.6.1 Thompson's group F

One group for which the question of amenability is non-trivial, open, and endlessly peculiar, is Thompson's group F. It was first used to construct finitely-presented groups with unsolvable word problems (Burillo, 1999), but has since found a variety of other applications. A useful survey was given by Cannon, Floyd, and Parry (1994). One intuitive ways to visualise elements of Thompson's group F is to use the rectangle diagrams invented by W. Thurston in 1975.

Thompson's group F is known by the following two presentations, but has a variety of other interesting interpretations.

Definition 2.32 *Thompson's group* F is given by

(i) $F = \left\langle x_0, x_1, x_2, \ldots \middle| x_k^{-1} x_n x_k = x_{n+1} \text{ for all } k < n \right\rangle$, or alternatively,

(ii) $F = \left\langle a, b \middle| [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2] \right\rangle$.

The group F is finitely presented, and has exponential growth. It contains an Abelian free group of infinite rank and therefore is not hyperbolic. Most particularly, F does *not* contain a subgroup isomorphic to $\mathbb{F}_2$, yet is not in any of the classes of groups known to be amenable (elementary, solvable, Abelian, and so on). Attempting to answer the amenability question with e.g. Theorem 2.30(viii) winds up in a vaguely Hofstadterian strange loop:[6] all finite-index subgroups of F are either isomorphic to F or a non-split extension of a finite cyclic group with F (Bleak and Wassink, 2007). By the above infinite presentation, it is clear that F is effectively an infinitely-iterated HNN extension, and is isomorphic to its own HNN extension. Finally, F an interesting example since there is no intuitive principle on which F is either obviously amenable, or obviously non-amenable.

To illustrate the problem a bit further, let's assume F is amenable and has a finitely-additive probabiliy measure μ. In the infinite presentation above, let F_n denote the subgroup of F consisting of words in the generators numbered from n and above, i.e. $F_n = \langle x_n, x_{n+1}, \ldots \rangle$. Clearly $F = F_0$, $F_n \cong F$ for any n, and $F_n \subset F_{n-1} \subset \cdots \subset F_2 \subset F_1 \subset F$. Furthermore we can work from F_n back up to F by HNN-extending by x_{n-1}, x_{n-2}, etc. It is simple to show that $\mu(F_1) = 0$ as follows. For any $n \geq 1$, note that

$$F_1 = F_n \cup (F_1 \backslash F_n)$$

and also

$$F_n = x_0^{1-n} F_1 x_0^{n-1}.$$

Now

$$\begin{aligned}
\mu(F_1) &= \mu(F_n) + \mu(F_1 \backslash F_n) \\
&= \mu\left(x_0^{1-n} F_1 x_0^{n-1}\right) + \mu(F_1 \backslash F_n) \\
&= \mu(F_1) + \mu(F_1 \backslash F_n)
\end{aligned}$$

thus $\mu(F_1 \backslash F_n) = 0$, for all $n \geq 1$.

Now, $F_3 = x_0^{-2} F_1 x_0^2$ and $\mu(F_1 \backslash F_3) = 0$. $x_1 F_3 \subset F_1 \backslash F_3$, since x_1 is not a generator

[6] See the immortal *Gödel, Escher, Bach* by Hofstadter (1979). Hofstadter's Law was indispensable in the production of the present work.

of F_3. Therefore

$$0 = \mu(F_1 \backslash F_3) \geq \mu(x_1 F_3) = \mu(F_3),$$

and hence $\mu(F_1) = \mu(F_1 \backslash F_3) + \mu(F_3) = 0$.

Hence $\mu(F \backslash F_1) = 1$. By repeatedly conjugating with x_0, increasing subsets of $F \backslash F_1$ can be equidecomposed into subsets of F_1, but there will always be some elements left over. Note that $F \backslash F_1$ is not merely $\langle x_0 \rangle$—it contains every element that requires the generator x_0 in its expression (e.g. $x_0 x_1 \in F \backslash F_1$, but not $x_0^{-1} x_1 x_0 = x_2 \in F_1$). Therefore $F \backslash F_1$ is still a complicated place to have all the mass, and so there is not likely to be a simple set of working in this fashion to show that $\mu(F) = 0$.

Prior to the proofs of the now-known counterexamples to the $AG = NF$ conjecture, there was considerable hope that F might be non-amenable, and therefore a counterexample. If, on the other hand, F is amenable, then it would be a counterexample to the conjecture that $AG = EG$ when restricted to the class of finitely presented groups.

Recent research continues to be conflicted about whether or not F is amenable. Akhmedov et al. (2009) answered in the negative and Shavgulidze (2009) in the positive, within a short time of one another, however both proofs were found to have unfixable flaws. A preprint of Moore (2012) claimed to answer in the positive, taking advantage of a connection with structural Ramsey theory devised earlier (Moore, 2011). This fomulation corresponds to a significant weakening of the Følner criteria. However an unfixable flaw was also found in this work by Ahkmedov. A running commentary on this saga is on the website of Calegari (2009).

It might not be possible to find a proof, but there may still be evidence hinting that F either is or is not amenable. One approach, attempted by Burillo, Cleary, and Wiest (2007) and later by Elder, Rechnitzer, and Wong (2011), was to estimate the cogrowth rate empirically. If the cogrowth rate of the rank 2 presentation of F appears to converge to 3, this would be significant evidence that F was amenable. Conversely, if it converges to some other value, then that would be evidence that F is not. As a computational problem, this approach is challenging: the cogrowth of F is clearly exponential, so in the worst case would involve individually checking an exponentially-increasing set of elements. Recently, Elder et al. (2013) announced that an improved algorithm using a Metropolis-style Markov chain technique was able to estimate the cogrowth accurately for many known finitely-presented examples. Their program indicated that F is not amenable.

	Examples	Source
Amenable	Finite	Wagon (1993)
	Abelian	Wagon (1993)
	Virtually solvable	Wagon (1993)
	Subexponential	Wagon (1993)
	Other elementary	Wagon (1993)
		Grigorchuk (1996)
Non-amenable	$\mathbb{F}_2$	Example 2.4,
		Wagon (1993) p.147,
		Paterson (1988) p.6
	Ol'shanskii first example	Wagon (1993) p.14
	$B(2, 665)$	Adian (1983)
	Ol'shanskii second example	Ol'shanskii and Sapir (2002)
Not known	Thompson's group F	*Evidence against:*
		Elder et al. (2013).

Table 2.4: Important examples of finitely-generated groups.

2.7 Topological group amenability

Suppose that G is a locally-compact Hausdorff group (a property supported by the discrete topology, and therefore by discrete groups). We have, then, the option of taking the topology of G into account. The theory exists for non-Hausdorff and non-locally-compact groups, but is, according to Wagon (1993), more coherent in the class of locally-compact groups.

The σ-algebra of Borel sets is the σ-algebra generated by the collection of all open sets. Alternatively, and usually equivalently, one can define the Borel sets to be those generated by compact sets. Note that, for an arbitrary locally-compact Hausdorff topology, not all subsets of the group are necessarily going to be Borel. Then a very reasonable approach is as follows.

Definition 2.33 (Topological amenability–I) A locally-compact and Hausdorff group G is *(topologically) amenable* if there is a finitely-additive, left invariant measure on the Borel sets of G, with total measure 1. (Wagon, 1993)

Since in the discrete topology all sets are open, so all sets are Borel, and thus topological amenability and the previously-discussed *discrete* amenability are the same when considering the discrete topology. Topological amenability thus goes in an interesting direction when considering coarser topologies than the discrete topology, although any group is amenable under the trivial topology.

By Haar measure theory, there exists a non-trivial left-invariant countably-additive measure λ defined on the (compactly-generated) Borel sets with total measure 1. As a result, any compact group is topologically amenable.

However, the "right" approach (Runde, 2002, p17) to topological amenability is a generalisation of Definition 2.7. The Haar measure λ and Borel algebra on G form the basis for Lebesgue integration and the usual function space norms. As a result, the function spaces $L^1(G), L^\infty(G)$ and so on make sense. These are the continuous analogues of $\ell^1(G), \ell^\infty(G)$ and so on, completed under the norm, and modulo the λ-null sets.

Definition 2.34 (Topological amenability–II) As before, a linear functional $m \in L^\infty(G)^*$ is called a *mean* if $m(\chi_G) = 1$ and $m(f) \geq 0$ for all $f \in L^\infty(G), f \geq 0$. A locally-compact group G is (topologically) amenable if there exists a left-invariant mean.

$$\begin{tikzcd}
P(G) \arrow[r, "\text{subset}"] \arrow[dd, "\wedge"] & L^1(G) \arrow[d, "\text{dual}"] & \\
& L^1(G)^* \arrow[r, "\text{equiv.}"] & L^\infty(G) \arrow[d, "\text{dual}"] \\
P(G)^\wedge \arrow[r, "\text{w* dense in}"] & \mathfrak{M}(G) \arrow[r, "\text{subset}"] & L^\infty(G)^*
\end{tikzcd}$$

Figure 2.2: An overview of the sets mentioned in the functional approach to topological amenability. We find a mean in $\mathfrak{M}(G)$ as a weak* cluster point of a sequence in $P(G)^\wedge$, and can use various analytic machinery along the way.

Similarly, the case of real-valued functions $f \in L^\infty(G)$, it can be shown that any mean m satisfies

$$\operatorname{ess}\inf_{x\in G} f(x) \leq m(f) \leq \operatorname{ess}\sup_{x\in G} f(x)\,.$$

Let $\mathfrak{M}(G) \subset L^\infty(G)^*$ denote the space of means. Paterson (1988) asserts that $\mathfrak{M}(G)$ is difficult to work with, however, there is a more convenient subspace $P(G)^\wedge$, from which invariant means can be found as cluster points, and is constructed as follows.

Lemma 2.35 Let

$$P(G) = \left\{ f \in L^1(G) : f \geq 0, \int f d\lambda = 1 \right\}$$

and let $^\wedge$ denote the canonical embedding of $L^1(G)$ in its second dual, i.e. $^\wedge$: $L^1(G) \to L^1(G)^{**} \equiv L^\infty(G)^*$ given by $\hat{f}(\phi) = \phi(f)$. $P(G)^\wedge$ is weak*-dense in $\mathfrak{M}(G)$. (Not shown here.)

Finding a mean then amounts to finding a sequence from $P(G)^\wedge$ that has a mean with the desired property as a weak* cluster point.

2.7.1 Topological Følner criterion

The topological equivalent of the Følner criterion essentially and mundantely replaces finite sets with compact sets and cardinality with Haar measure.

Theorem 2.36 (Paterson, 1988) A locally-compact group G is amenable if, and only if, for every $\epsilon > 0$, compact set C, and $x \in C$, there exists a compact set F such that

$$\frac{\lambda\left(xF \triangle F\right)}{\lambda\left(F\right)} < \epsilon.$$

(Not shown here.)

Chapter 3

Amenability and Banach algebras and C*-algebras

(…Oh my!)

Much modern amenability theory is explored with a focus on Banach and C* algebras.

The sequence/function space $\ell^1(G)$ or $L^1(G)$ (as the case may be) is a Banach space. By adding the convolution operation, it becomes a Banach algebra (the *semigroup algebra*). In 1972, B. E. Johnson noted that it is possible to define on this algebra a condition which characterises the amenability of the discrete group G, but described only with mechanisms related to that algebra. Group amenability, ostensibly a property involving the existence of a functional that is loosely related to the group, is thus encoded with the techniques of cohomology and Banach algebra theory. Therefore amenability extends naturally to arbitrary Banach algebras, so, it is sensible to speak of *amenable* Banach algebras, and some results from cohomology theory (regarded as powerful and useful) can be brought to bear on amenability.

3.1 Banach algebra amenability

Recall that a *module* over a *ring* is analogous to a *vector space* over a *field*—the ring/field describes the scalars, which act on the module/vector space. Since algebras are examples of rings, we might consider modules over algebras. (An algebra, which is also a vector space, should not be confused with the module being acted upon by the algebra.)

Definition 3.1 For the algebra $\mathfrak{A}$, a space E is a *left $\mathfrak{A}$-module* if the left action of $\mathfrak{A}$ on E is bilinear and compatible with the ring multiplication, i.e. $a \cdot (b \cdot x) = ab \cdot x$ for all $a, b \in \mathfrak{A}$, $x \in E$. Similarly, *right* $\mathfrak{A}$-module. A $\mathfrak{A}$-*bimodule* is obviously a left and right $\mathfrak{A}$-*module*, but also requires $(a \cdot x) \cdot b = a \cdot (x \cdot b)$.

A Banach space E is called a *left Banach $\mathfrak{A}$-module* when E is a left $\mathfrak{A}$-module and there exists a $K > 0$ such that

$$\|a \cdot x\| \leq K \|a\| \|x\|$$

that is, the representation of $\mathfrak{A}$ on E is norm-continuous. We can fix $K = 1$ by allowing equivalent norms.

Lemma 3.2 If E is a Banach $\mathfrak{A}$-bimodule, then the dual, E^*, is also a Banach $\mathfrak{A}$-bimodule.

Definition 3.3 A bounded linear map $D : \mathfrak{A} \to E$ is called a *derivation* if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad \text{for all} \quad a, b \in \mathfrak{A}.$$

The name “derivation” is hence a nod to the derivative operation from undergraduate calculus: if a, b were functions and D was the derivative operator, then the equation above is, analogously, the Leibniz rule.

Definition 3.4 A derivation is *inner* if it is of the form

$$\mathrm{ad}_x(a) := a \cdot x - x \cdot a \quad \text{for some} \quad x \in E.$$

The space of *all* derivations is denoted $\mathcal{Z}^1(\mathfrak{A}, E)$ and the space of all *inner* derivations is denoted $\mathcal{B}^1(\mathfrak{A}, E)$.

The *first Hochschild cohomology group* (of $\mathfrak{A}$ with coefficients in E), denoted $\mathcal{H}^1(\mathfrak{A}, E)$ is defined by

$$\mathcal{H}^1(\mathfrak{A}, E) := \mathcal{Z}^1(\mathfrak{A}, E)/\mathcal{B}^1(\mathfrak{A}, E).$$

Definition 3.5 (Amenable Banach algebras) A Banach algebra $\mathfrak{A}$ is called *amenable* if $\mathcal{H}^1(\mathfrak{A}, E^*) = \{0\}$ for every Banach $\mathfrak{A}$-bimodule E. (Runde, 2002, p43)

Said another way, $\mathfrak{A}$ is amenable if for every dual Banach bimodule E^*, all derivations $D : \mathfrak{A} \to E^*$ are inner derivations. The terminology is justified by the following result.

Theorem 3.6 (Johnson's Theorem) For a locally-compact group G, the convolution algebra $L^1(G)$ is amenable if and only if G is amenable. (Not shown here.)

3.2 C*-algebraic amenability

The C*-algebras are an important and widely-studied class of Banach algebras. A range of C*-algebraic theorems can be applied to amenability. Before discussing amenable Banach algebras further it is necessary to introduce more concepts.

Definition 3.7 A *Banach $*$-algebra* is a Banach algebra $\mathfrak{A}$ together with an involution $* : \mathfrak{A} \to \mathfrak{A}$ that is an isometry, and such that $(ab)^* = b^*a^*, (a+b)^* = a^* + b^*, (\lambda a)^* = \bar{\lambda} a^*$.

A *C*-algebra* is a Banach $*$-algebra that satisfies the *C* condition*:

$$\|x^*x\| = \|x\| \, \|x^*\| \, .$$

Remark 3.8 The isometry of the involution operation follows from the C* condition. The C* condition is equivalent to the *B* condition*, which is $\|xx^*\| = \|x\|^2$. Examples of C*-algebras include the $n \times n$ matrices over $\mathbb{C}$ with the operator norm, the space of bounded linear operators over a Hilbert space $\mathcal{H}$, denoted $\mathcal{B}(\mathcal{H})$, and von Neumann algebras (also known as W*-algebras).

Amenability for the special case of C*-algebras depends on the concept of a tensor product, which is breifly recapped as follows.

Definition 3.9 The *tensor product* of two linear spaces A and B, denoted $A \otimes B$, is the unique linear space (up to isomorphism) and associated homomorphism $\tau : A \times B \to A \otimes B$ such that for any other linear space F and bilinear homomorphism $V : A \times B \to F$, there exists a bilinear homomorphism $\tilde{V}$ such that the following diagram commutes:

$$\begin{tikzcd} A \times B \arrow[d, "\tau"] \arrow[dr, "V"] & \\ A \otimes B \arrow[r, "\tilde{V}"] & F \end{tikzcd}$$

That is, every bilinear map factors uniquely through the tensor product: the tensor product is the "free-est" bilinear space, and every other is "less free".

The tensor product of spaces is associative. What do tensor product spaces look like? They may be constructed as follows. Elements of tensor products of spaces are charactarised as being linear combinations of *elementary tensors*. Elementary tensors in $E_1 \otimes E_2 \otimes \cdots \otimes E_n$ are tensors of the form

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n := \tau(x_1, x_2, \ldots, x_n).$$

Thus any tensor t in a tensor product space might look somewhat like

$$t = \sum_{i=1}^{m} \lambda_i \left[x_1^{(i)} \otimes x_2^{(i)} \otimes \cdots \otimes x_n^{(i)} \right].$$

A tensor product of two Banach spaces is not often going to be a Banach space itself. The usual approach is to complete with respect to an appropriate norm. Not every norm, when completed, will yield a useful Banach space, therefore one useful condition is as follows.

Definition 3.10 A *cross norm* is a norm $\|\cdot\|$ defined on a tensor product $A \otimes B$ such that $\|a \otimes b\| = \|a\| \, \|b\|$ for all $a \in A, b \in B$.

There are two particular cross norms which are pertinent to these discussions.

Definition 3.11 Consider the tensor product $A \otimes B$.

(i) The *injective* cross norm, denoted $\|\cdot\|_\epsilon$, is the smallest cross norm, and is given by

$$\|x\|_\epsilon := \sup \{ |\{a^* \otimes b^*\}(x)| : a^* \in A^*, b^* \in B^*, \|a^*\| = 1 = \|b^*\| \}$$

for all $x \in A \otimes B$.

(ii) The *projective* cross norm, $\|\cdot\|_\pi$, is the largest cross norm, and is given by

$$\|x\|_\pi := \inf \left\{ \sum_{i=1}^{n} \|a_i\| \, \|b_i\| : x = \sum_{i=1}^{n} a_i \otimes b_i \right\}$$

for all $x \in A \otimes B$ (that is, the infimum is taken over all the finite sums of elementary tensors summing to x).

Completing a tensor product with respect to these yield the *injective* and *projective tensor products*, denoted here using $\check{\otimes}$ and $\hat{\otimes}$, respectively.

Definition 3.12 A C*-algebra $\mathfrak{A}$ is *nuclear* if, for any other C*-algebra $\mathfrak{B}$, the injective and projective cross-norms coincide on $\mathfrak{A} \otimes \mathfrak{B}$.

Theorem 3.13 A C*-algebra $\mathfrak{A}$ is amenable if and only if it is *nuclear*. (Not shown here. See Pier, 1984)

3.3 Banach algebras again

Amenable Banach algebras are characterised as those having approximate diagonals and having virtual diagonals. This relies on the projective tensor product introduced above.

Definition 3.14 Let $\mathfrak{A}$ be a Banach algebra. Define the *diagonal operator* $\Delta_{\mathfrak{A}} : \mathfrak{A}\hat{\otimes}\mathfrak{A} \to \mathfrak{A}$ by setting

$$\Delta_{\mathfrak{A}}(a \otimes b) := ab \quad \text{for all } a \otimes b \in \mathfrak{A}\hat{\otimes}\mathfrak{A}.$$

Definition 3.15 (Runde, 2002, p44) Let $\mathfrak{A}$ be a Banach algebra. The element $M \in (\mathfrak{A}\hat{\otimes}\mathfrak{A})^{**}$ is a *virtual diagonal for* $\mathfrak{A}$ if, for all $a \in \mathfrak{A}$,

$$a \cdot M = M \cdot a \quad \text{and} \quad a \cdot \Delta_{\mathfrak{A}}^{**} M = a.$$

A bounded net $(m_\alpha)_\alpha$ in $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ is an *approximate diagonal for* $\mathfrak{A}$ if, for all $a \in \mathfrak{A}$,

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0 \quad \text{and} \quad a \cdot \Delta_{\mathfrak{A}} m_\alpha \to a.$$

Theorem 3.16 (Runde, 2002, p45) These three conditions are equivalent for the Banach algebra $\mathfrak{A}$:

(i) $\mathfrak{A}$ is amenable.

(ii) $\mathfrak{A}$ has an approximate diagonal.

(iii) $\mathfrak{A}$ has a virtual diagonal.

(Not shown here.)

3.4 The weak containment property

C*-algebras are characterised as algebras of operators, and are also coupled to representation theory of groups. So it is that amenability goes full circle: from groups to Banach algebras to C*-algebras to representations and back to groups.

Definition 3.17 For a locally-compact group G, the representation π is *weakly contained in the representation* τ if all positive definite functions associated with π are uniform limits on the compact subsets of G of sums of positive functions associated with τ (Pavel, 2007).

Alternatively: for representations $\pi, \tau : G \to \mathcal{U}(\mathcal{H})$, π is weakly contained in τ (denoted $\pi \preceq \tau$) if for every $\xi \in \mathcal{H}$, finite $F \subseteq G$, and $\epsilon > 0$ there exists $\eta_1, \ldots, \eta_n \in \mathcal{H}$ such that

$$\left| \langle \pi(g)\, \xi, \xi \rangle - \sum_{i=1}^{n} \langle \tau(g)\, \eta_i, \eta_i \rangle \right| < \epsilon$$

for all $g \in F$ (Peterson, 2011).

We say that G has the *weak containment property* when each irreducible unitary representation is weakly contained in the left regular representation (Pavel, 2007). Briefly, the recall that the left regular representation, π_2, is given by:

$$\pi_2(g)\xi := \sum_{t \in G} \xi_t \mathbf{e}_{gt} = \sum_{t \in G} \xi_{g^{-1}t} \mathbf{e}_t$$

for all $\xi = \sum_{t \in G} \xi_t \mathbf{e}_t \in \ell^2(G)$.

In representation theory of groups, the following result is well-known.

Theorem 3.18 The following are equivalent for a group G.

(i) G is amenable.

(ii) G has the weak containment property.

(iii) The trivial representation of G is weakly contained in the left regular representation.

(iv) The algebras $C^*(G)$ and $C^*_r(G)$ are $*$-isomorphic.

All but the last condition have been adequately defined above. What are $C^*(G)$ and $C^*_r(G)$? A more complete account will be given in the next chapter (§4.4.1), in the generality of inverse semigroups. Briefly, however, $C^*(G)$ is the *C*-algebra of* G, and is more explicitly called the C*-enveloping algebra of $\ell^1(G)$. $C^*(G)$ is defined as the completion of $\ell^1(G)$ under the supremum norm over all representations (of G in some implied Hilbert space $\mathcal{H}$). $C_r^*(G)$ is the *reduced C*-algebra of* G, and is the norm closure of $\pi_2(\ell^1(G))$.

Chapter 4

Amenability and Semigroups

Amenability is easily generalisable to semigroups, but results vary depending upon the choice of generalisation. Various nice theorems for groups, relating amenability to other properties, do not hold for semigroups.

4.1 Semigroups with finitely-additive measures

Recall Definition 2.1. At no point in the definition was anything particularly "groupish" involved—no inverses, no identity element (or even associativity for that matter). The definition *could* apply immediately to any semigroup, using the natural left and right actions (left and right regular representations) of a semigroup on itself.

Definition 4.1 A semigroup S is *left measurable* if there exists a finitely-additive measure $\mu : \mathcal{P}(S) \to [0, 1]$ that is left invariant and has total measure 1 (Klawe, 1977; Paterson, 1988). Compare with Definition 2.1.

A left measurable semigroup S has the following properties, attributed to J. R. Sorenson (Klawe, 1977, p103).

(i) A homomorphic image of S is not necessarily left measurable.

(ii) A left ideal of S is not necessarily left measurable, but a right ideal must be left measurable.

Furthermore,

(i) A finite direct product of left measurable semigroups is left measurable.

(ii) A directed union of left measurable semigroups is left measurable.

Compare these with Theorem 2.30.

To motivate what will come later, the following result demonstrates why measurability is not a definition of "semigroup amenability" in common use.

Theorem 4.2 A non-trivial semigroup with zero S *cannot* be left or right measurable.

Proof This proof is similar to the one given by van Douwen (1992). For the semigroup S with zero 0 and finitely-additive measure μ,

$$\begin{aligned}
1 &= \mu(S) && \because \text{total measure } 1 \\
&= \mu(0S) && \because \text{left-invariance} \\
&= \mu(\{0\}) \\
&= \mu(0(S\setminus\{0\})) \\
&= \mu(S\setminus\{0\}) && \because \text{left-invariance} \\
&= \mu(S) - \mu(\{0\}) && \because \text{finitely additive} \\
&= 1 - 1 = 0,
\end{aligned}$$

a contradiction. This also holds for right-invariance. □

Here are some reasons that, beyond mere tradition, left measurability is not often considered on semigroups:

- Zero elements are a common feature of many semigroups, and is utterly unsurprising that the behaviour of a zero element is necessarily "volume-destroying". Therefore, we would expect that whatever definition is used should account for that behaviour, and Theorem 4.2 demonstrates that Defintion 2.1 is deficient in this regard.
- As yet, there appears to be no satisfying weakening of the three conditions.
- Finitely-additive measure theory is less sophisticated and powerful than the definition that uses means. Given the power of the more sophisticated alternative, it is not surprising that the definition from Day (1957) is preferred.

Left measurable semigroups will return later on.

4.2 Semigroups with means

Now recall Definition 2.7. This formulation was given in Day (1957) directly in terms of semigroups, and is the most popular. Means and left-invariance are exactly as previously defined.

Definition 4.3 A semigroup S is *(classically) left-amenable* if there exists a left-invariant mean on S, similarly for right- and bi-amenable.

In spite of the lack of inverses and cancellativity in a general semigroup S, the two actions—the convolution action $*$ in $\ell^1(S)$ and the dual action $\cdot$ in $\ell^\infty(S)$—work almost as well as they do for groups, with some minor tweaking.

Suppose $\phi \in \ell^\infty(S)$ and $\hat{\phi} \in \ell^1(S)^*$ are equivalent under the usual identification of $\ell^1(S)^*$ and $\ell^\infty(S)$, i.e. such that

$$\hat{\phi}(f) = \langle f, \phi \rangle, \quad \text{and} \quad \phi(s) = \hat{\phi}\left(\chi_{\{s\}}\right)$$

for all $f \in \ell^1(S), s \in S$. On $\ell^1(S)$ we have the convolution of $f_1, f_2 \in \ell^1(G)$ given again as usual (note how this varies from convolution for a group):

$$(f_1 * f_2)(x) := \sum_{\substack{s,t \in S \\ st = x}} f_1(s) f_2(t).$$

Henceforth, a summation (or similar) over "$st = x$" shall mean "over all pairs $s, t \in S$ such that $st = x$." In particular, the convolution action carries over to $\ell^1(S)$ from the group case as well. For all $x \in S$,

$$\{s * f\}(x) = \sum_{t \in s^{-1}\{x\}} f(t).$$

While the $*$ action was well-defined for a group G on $\ell^\infty(G)$, here, the expression $s * \phi$ for $s \in S$ and $\phi \in \ell^\infty(S)$ is only well-defined in some cases, such as for the subspace $\ell^1(S)$.

Lemma 4.4 The left convolution action and the dual left action are duals of one another, i.e.

$$\langle \phi, s * f \rangle = \langle s \cdot \phi, f \rangle$$

for all $\phi \in \ell^\infty(S)$, $f \in \ell^1(S)$, and $s \in S$. (Compare with Lemma 2.9.)

Proof

$$\begin{aligned}\langle \phi, s * f\rangle &= \sum_{t\in S} \{s * f\}(t)\,\phi(t)\\ &= \sum_{t\in S}\left[\sum_{u\in s^{-1}\{t\}} f(u)\right]\phi(t)\\ &= \sum_{t\in S}\sum_{u\in s^{-1}\{t\}} f(u)\,\phi(su)\\ &= \sum_{u\in S} f(u)\,\phi(su)\\ &= \sum_{u\in S} f(u)\,\{s\cdot\phi\}(u)\\ &= \langle s\cdot\phi, f\rangle\end{aligned}$$

as required. □

So far so good—amenability of S is now (loosely) connected to the amenability of $\ell^1(S)$.

Lemma 4.5 Suppose S is a semigroup with a left zero z. S is right amenable because the mean δ_z given by $\delta_z(f) = f(z)$ for all $f \in \ell^\infty(S)$ is a right invariant mean for S. Furthermore, if S is left amenable then δ_z is the only left invariant mean.

Proof The mean δ_z is a right-invariant mean.

1. δ_z is a mean.
   PROOF: For all f, $\delta_z(f)$ must clearly be within the range of values taken by f, i.e.
   $$\inf_{x\in S} f(x) \le \delta_z(f) = f(z) \le \sup_{x\in S} f(x)\,.$$
2. δ_z is right-invariant.
   PROOF: For all $s \in S$,
   $$\begin{aligned}\delta_z(f\cdot s) &= \{f\cdot s\}(z)\\ &= f(zs)\\ &= f(z) \quad \because z \text{ is a left zero.}\end{aligned}$$

Now suppose S is left amenable with a left invariant mean m. $m(f) = m(z\cdot f)$ by left invariance, but $\{z\cdot f\}(t) = f(zt) = f(z)$ for all $t \in S$, and therefore $m(f) = f(z) = \delta_z(f)$. □

Since any two-sided zero is unique,

Corollary 4.6 Any semigroup S with zero 0 is amenable, and the invariant mean is unique.

Proof By Lemma 4.5, the mean m given by $m(f) = f(0)$ for all f is the unique left- and right-invariant mean for S. □

Together with Theorem 4.2, this demonstrates that Definition 2.1 and 4.3 are not equivalent. Unlike the case with groups, it seems that under Definition 4.3, a semigroup can be amenable *and* be scarcely any bigger than non-amenable subsemigroups and subgroups—merely adjoin a zero. Semigroup amenability is even more fickle than that, however.

Corollary 4.7 Suppose a semigroup S has two distinct left zeroes z_1 and z_2. Then S is *not* left amenable.

Proof Assume S is left amenable. Applying Lemma 4.5 to both z_1 and z_2, the only left invariant mean m is given by both $m(f) = f(z_1)$ and $m(f) = f(z_2)$ for all $f \in \ell^\infty(S)$. Any function f such that $f(z_1) \neq f(z_2)$ suffices to show a contradiction. □

4.2.1 Breakdown between definitions

It is reasonably easy to show that every left measurable semigroup is left amenable (Klawe, 1977, p102), but this may be the best we get between the two. Definition 4.3 sets a low bar.

No group is "trivially" amenable in the way that a semigroup with zero is. Consider the left action of a (semi)group element on an indicator function. If we are operating within a *group*, we may do the following:

$$(g\chi_A)(x) = \begin{cases} 1 & \text{if } gx \in A \\ 0 & \text{if } gx \notin A \end{cases} = \begin{cases} 1 & \text{if } x \in g^{-1}A \\ 0 & \text{if } x \notin g^{-1}A \end{cases} = \chi_{g^{-1}A}(x).$$

This is clearly not available to all semigroups. At minimum we require some substitute for $g^{-1}A$, and in fact if we attempt to deduce a finitely-additive measure μ from some left-invariant mean m we obtain for all $s \in S$ and $A \subseteq S$ the condition

$$\mu\left(s^{-1}A\right) = \mu(A)$$

where $s^{-1}A$ denotes the preimage, i.e. $s^{-1}A = \{t \in S : st \in A\}$. (Paterson, 1988, Exercise 0.32). On groups, this is the same as the usual measure invariance, since preimages coincide with inverses. On semigroups in general, it is *not* the same.

Example 4.8 Consider again a non-trivial semigroup with zero. Here the invariant mean is given by $m(f) = f(0)$. Suppose that we were to try to obtain an invariant finitely-additive measure μ from this definition by setting $\mu(A) = m(\chi_A)$. Then

$$\mu(A) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{if } 0 \notin A \end{cases} = |A \cap \{0\}|.$$

This μ is finitely additive (for disjoint A, B, 0 is in at most one of them) and has total measure 1 ($0 \in S$). However, $\mu(0(S\setminus\{0\})) = \mu(\{0\}) = 1$ and $\mu((S\setminus\{0\})) = 0$, so μ is not invariant (in that multiplication by 0 alters the value). Instead, it satisfies *preimage invariance* given by $\mu\left(s^{-1}A\right) = \mu(A) = \mu\left(As^{-1}\right)$. In particular, note that

$$0^{-1}A = \begin{cases} S & \text{if } 0 \in A \\ \emptyset & \text{if } 0 \notin A \end{cases}.$$

* * *

It seems that neither Definition 2.1 or Definition 2.7, when extended to semigroups, capture reasonable intuition about semigroups. Both definitions are fickle with respect to the presence or absence of zeroes. Why should $\mathbb{F}_2^0$ be considered amenable, when the sole reason is the presence of a zero?

4.3 Classical amenability results for semigroups

We temporarily suspend further arguments about different definitions for semigroups and continue with the traditional one given by Day (Definition 2.7). Here are some standard results.

4.3.1 Følner conditions and a theorem of Frey

Consider Theorem 2.16 and take, for example, $K = \{0\}$. Now $|FK\setminus F| = |\{0\}\setminus F| \leq 1$ for any F, and $1/|F|$ can be made arbitrarily small. Adding 0 to any K does not increase $|FK \backslash F|$ in the limit. Thus the presence of a zero does not affect whether the semigroup satisfies this Følner criterion.

On the other hand, consider Lemma 2.15. If $s = 0$ then $|sF_n \bigtriangleup F_n|$ becomes close to $|F_n|$ as n grows large, therefore there is no Følner sequence $\{F_n\}_{n \in \mathbb{N}}$, in the sense of Lemma 2.15, for $0 \in S$.

It is well-known that the Følner conditions do not produce the same results as the Day definition of amenability on semigroups (Paterson, 1988, p17). The question is then how they are related.

Følner's necessary condition (see §2.4.1) was generalised to semigroups by Alexander Frey in his 1960 thesis, and Namioka (1964) subsequently provided a simpler proof of the Følner-Frey theorem.

Theorem 4.9 (The Følner-Frey theorem) (Namioka, 1964)
Let S be a left-amenable semigroup. Then for any finite subset K of S and for any $\epsilon > 0$, there is a finite subset F of S such that for each $s \in K$,

$$|sF \backslash F| < \epsilon \, |F| \, .$$

This (left) Følner condition will be abbreviated (FC). (Proof not shown here.)

Here, however, the converse is not equivalent for semigroups in general, the obvious example being $\mathbb{F}_2^0$. Another possibly surprising counterexample to the converse are the finite non-amenable semigroups.

There are, however, many variations on the Følner condition, and (by now it should come as no surprise that) though they coincide for groups, they are not equivalent on semigroups. Some other identified Følner conditions on semigroups and the relationships between them were handily summarised by Yang (1987):

Definition 4.10 A semigroup S may satisfy one or more of the following (left) conditions:

(SFC) "Strong Følner Condition": For every $s \in S$ and $\epsilon > 0$ there exists a finite set F such that

$$|F \backslash sF| < \epsilon \, |F| \, .$$

(Argabright and Wilde, 1967). The conjecture that every left amenable semigroup satisfies SFC is the *Argabright-Wilde conjecture*, and is false (Klawe, 1977).

$$\begin{tikzcd}
 & & \text{(WFNC)} \arrow[dr, Rightarrow] \arrow[dd, Rightarrow] & \\
\text{(SFC)} \arrow[r, Rightarrow] & \text{(SFNC)} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{(WFC)} \arrow[dl, Rightarrow] \\
 & & \text{(A)} \arrow[r, Rightarrow] & \text{(FC)}
\end{tikzcd}$$

Figure 4.1: Implications that have been shown between the left Følner-type conditions and left amenability (A) of a semigroup. Additionally, (FC) $\not\Rightarrow$ (A) and (A) $\not\Rightarrow$ (WFC) in general. (Yang, 1987)

(WFC) "Weak Følner Condition": There is a number k, $0 < k < 1$, such that for any finite list of n elements $a_1, \ldots, a_n$ of S (not necessarily distinct) there exists a finite set F such that

$$\frac{1}{n}\sum_{i=1}^{n} |F \backslash a_i F| \le k\,|F|\,.$$

(WFNC) "Weak Følner-Namioka Condition": There is a number k, $0 < k < 1$, such that for any finite list of $2n$ elements $a_1, \ldots, a_n; a'_1, \ldots, a'_n$ of S (not necessarily distinct) there exists a finite set F such that

$$\frac{1}{n}\sum_{i=1}^{n} |a_i F \backslash a'_i F| \ge k\,|F|\,.$$

(SFNC) Same as (WFC), except $k < \frac{1}{2}$.

Lemma 4.11 For left cancellative semigroups, (FC) = (SFC).

Proof For any semigroup S and finite set $F \subseteq S$,

$$2\,|F \backslash sF| \ge |F \bigtriangleup sF| \ge 2\,|sF \backslash F|\,.$$

If S is cancellative then $|sF| = |F|$, and furthermore the above inequality is saturated. $\square$

The relationship between these conditions and left amenability (A) for general semigroups is summarised in the diagram of implications in Figure 4.1.

4.3.2 Semigroups and finite means

An important result for finite means on semigroups by Day (1957) is given below. It was emphasised as an alternative definition of amenability by Namioka (1964) in order to simplify the proof of the Følner-Frey theorem. As a consequence, the analogue of Theorem 2.11 for semigroups can be shown in the same way.

Definition 4.12 Let $\Phi(S)$ denote the subset of $\ell^1(S)$ consisting of finite means. Recall (Definition 2.7) that an element $\nu \in \ell^1(S)$ is a *finite mean* if $\nu(s) \geq 0$ for all $s \in S$, supp(ν) is finite, and $\|\nu\|_1 = 1$. The nomenclature is justified because, for every finite mean ν, there is a corresponding mean m_ν given by

$$m_\nu(\phi) := \langle \nu, \phi \rangle \quad \text{for all } \phi \in \ell^\infty(S).$$

It is known that $\Phi(S)$ is the convex hull[1] of S within $\ell^1(S)$.

Theorem 4.13 A semigroup S is right amenable if, and only if, for every $s \in S$ there exists a net $\{f_\gamma\}$ in $\Phi(S)$ for which the net $\{f_\gamma * s - f_\gamma\}$ converges to 0 weakly in $\ell^1(S)$.

A semigroup S is *strongly* right amenable if, and only if, instead of weak convergence to 0, we have

$$\lim_\gamma \|f_\gamma * s - f_\gamma\|_1 = 0.$$

Strong right amenability is equivalent to right amenability. (Day, 1957; Namioka, 1964)

Proof For the first part, use the canonical embedding of $\ell^1(S)$ in its second dual, $\ell^\infty(S)^*$.

Secondly, let $E = (\ell^1(S))^S$ and $T : \ell^1(S) \to E$ be given by $T(f)(s) = f * \chi_{\{s\}} - f$. The argument then hinges on the fact that the closures of $T(\Phi(S))$ under the two different topologies on E (the weak topology and the product of norm topologies) are identical, and therefore 0 is in one closure iff it is in the other. □

Note that in particular, S is *not* right amenable when there is some continuous linear functional on E strongly separating 0 and $T(S)$. (Namioka, 1964)

[1] Recall that a *convex hull* is the smallest convex superset of a given set.

4.3.3 More standard results

Theorem 4.14 If a semigroup S is left amenable and right amenable, then it is bi-amenable: there is a two-sided invariant mean m. (Day, 1957, p515)

Proof The two-sided mean is a product of the left and right invariant means, and is exactly the same idea as Lemma 2.3. The product of two means corresponds to the "idea of Arens," that is, a product $\odot$ defined on the second dual of a Banach algebra in a three-step process. For our purposes, if m and n are two means for S, respectively, then the following defines $m \odot n$:

$$\{m \odot n\}(f) := m(n \odot f) \quad \text{for all } f \in \ell^\infty(S),$$

wherein $n \odot f \in \ell^\infty(S)$ is defined by

$$\{n \odot f\}(x) := n(x \cdot f) \quad \text{for all } x \in S.$$

If we accept $x \mapsto n(x \cdot f)$ as a substitute notation for the function $n \odot f$, then this can be defined in one step as

$$\{m \odot n\}(f) = m(n \odot f) = m(x \mapsto n(x \cdot f)).$$

Thus it is simple to check that if m is left invariant, and n is right invariant, the $m \odot n$ is both left and right invariant. On the left,

$$\begin{aligned} \{m \odot n\}(s \cdot f) &= m(n \odot (s \cdot f)) \\ &= m(s \cdot (n \odot f)) \\ &= m(n \odot f) = \{m \odot n\}(f) \end{aligned}$$

since m is left invariant, and on the right,

$$\begin{aligned} \{m \odot n\}(f \cdot s) &= m(x \mapsto n(x \cdot (f \cdot s))) \\ &= m(x \mapsto n((x \cdot f) \cdot s)) \\ &= m(x \mapsto n(x \cdot f)) \quad \because n \text{ right invariant} \\ &= \{m \odot n\}(f), \end{aligned}$$

as required. □

The example of adjoining a zero to obtain an amenable semigroup from a non-amenable one was known to Day (1957), but it was remarked with little fanfare. He also noted that,

Lemma 4.15 If S is a left-amenable semigroup with some left-invariant mean m, and T is a subsemigroup with $m(T) > 0$, then T is also left-amenable. (Day, 1969)

We have seen examples of finite semigroups which are not amenable. The question of when exactly a finite semigroup is amenable was resolved by Rosen in 1954 (Day, 1957).

Theorem 4.16 A finite semigroup S is left amenable if and only if it has a unique minimal right ideal, and *vice-versa*. S amenable if and only if it has a unique minimal left ideal and a unique minimal right ideal. In this case the two coincide and the ideal is a finite group, so S is endowed with the counting measure. (Not shown here.)

For groups, left amenability implies right amenability and vice-versa, but for semigroups, there are semigroups which are one but not the other. For example, let S be a right zero semigroup, $s \in S$, $f \in \ell^\infty(S)$. $(sf)(x) = f(sx) = f(x)$ so any mean witnesses S as left amenable, however $(fs)(x) = f(s)$, a constant, therefore a mean $m(f) = f(s)$, so any non-constant f demonstrates that all means on S are not right invariant, so S cannot be right amenable.

Another theorem of Day (1942), based on earlier work by Agnew and Morse (1938), is that Abelian semigroups are amenable. A more sophisticated but easy proof of this fact is as a simple consequence of the Markov-Kakutani fixed point theorem, as shown below (Paterson, 1988, p16).

Theorem 4.17 All (locally compact) Abelian semigroups are amenable.

Proof We desire for each Abelian semigroup S some left- or right-invariant mean m. But left-invariance is a kind of fixed-point property: a mean m is left-invariant if it is a fixed point of every left action $m \mapsto sm$ for $s \in S$. If S is Abelian, then almost by definition the left actions are commutative. The left actions are also affine, continuous mappings of the space $\mathfrak{M}(S)$ into itself, i.e. the space of means on S, and $\mathfrak{M}(S)$ is a compact convex subset of $L^\infty(S)^*$.

Applying the Markov-Kakutani theorem to S (now a commuting family of continuous affine maps of $\mathfrak{M}(S)$ to itself), there is at least one common fixed points in $\mathfrak{M}(S)$: the left-invariant mean, as required. □

Luthar (1962) showed a characterisation of the Abelian semigroups with unique invariant means. In particular, a discrete Abelian semigroup has a unique invariant mean if, and only if, it has a finite ideal. Likewise, a topological Abelian semigroup has a unique invariant mean if, and only if, it has a compact ideal.

4.3.4 Cancellative and reversible semigroups

In the group case, every subgroup of an amenable group is amenable. Obviously there are many cases of amenable semigroups containing non-amenable groups or semigroups—again, consider adjoining a zero to a non-amenable group—so this is out. However, it was shown by Frey in his thesis of 1960 that:

Theorem 4.18 Let S be a cancellative semigroup such that S contains no free subsemigroup on two generators. If S is left amenable, then every subsemigroup of S is left amenable. (Donnelly, 2012) $\square$

Therefore the rôle of FS_2 in semigroup amenability would be similar to that of $\mathbb{F}_2$ in group amenability. Donnelly (2012) then goes on to show its more important to look at it from the perspective of the subsemigroup:

Theorem 4.19 Let S be a cancellative semigroup. Let T be a subsemigroup of S such that T does not contain a free subsemigroup on two generators. If S is left amenable, then T is left amenable. (Donnelly, 2012) $\square$

The converse is false. For example, FS_2 is cancellative and non-amenable but contains amenable subsemigroups (e.g. FS_1).

A *left reversible* semigroup is one in which $aS \cap bS \neq \emptyset$ for all elements $a, b \in S$, or equivalently, that every pair of right ideals has non-empty intersection. Similarly, *right* reversible for $Sa \cap Sb \neq \emptyset$ and left ideals.

Theorem 4.20 Every left amenable semigroup is left reversible.

Proof Here we use the shorthand $m(A)$ in place of $m(\chi_A)$. If m is a left-invariant mean, then $m(R) = 1$ for any right ideal R. For right ideals R_1, R_2, $m(R_1) = 1 = m(R_2)$ so $m(R_1 \cap R_2) = 1$ and therefore $R_1 \cap R_2 \neq \emptyset$. (Paterson, 1988) $\square$

For example, any semigroup with zero (being amenable) has at least $0 \in aS \cap bS$ for any a, b. Of course, nontrivial semigroups with a zero are not cancellative.

Reversibility and cancellativity are related to the embeddability of a semigroup within a group. If a semigroup fails to be cancellative, then it cannot be embedded in a group, but it is not sufficient in general. For the class of commutative semigroups, cancellativity is sufficient for embedding in a group. Another case where cancellativity becomes sufficient for embedding in a group is left reversibility. The following theorem was originally shown for rings and fields, but is useful enough to be famous in the literature on semigroups.

Theorem 4.21 (Ore's Theorem for semigroups) Let S be a cancellative semigroup. If S is left reversible, then S is embeddable in the group $G(S) = \{st^{-1} : s, t \in S\}$. (Not shown here. See Clifford and Preston, 1967, p35, and Paterson, 1988, p36.)

The $G(S)$ in the above theorem is called the *group of fractions*. Another way to obtain $G(S)$ is by $G(S) = \langle S | xy = z : x, y, z \in S \rangle$. The group of fractions is important, as it is universal for all homomorphisms from S to any group. There is a natural map from S to the generators of $G(S)$, and if that map is not an injection, then S won't be embeddable in any group.

Corollary 4.22 Every left amenable cancellative semigroup S is embeddable in a group.

However, more can be said of the left amenability of left reversible semigroups in general.

Theorem 4.23 Let S be a left reversible semigroup, and let $\approx$ be the congruence on S defined by setting, for all $x, y \in S$,

$$x \approx y \Leftrightarrow \exists s \in S \text{ such that } xs = ys.$$

Note that $S/\approx$ is right cancellative. Then S is left amenable if, and only if, $S/\approx$ is left amenable. (Paterson, 1988, p35)

This reduces the question of the left amenability of a left reversible semigroup to the subclass of right cancellative semigroups. If we can get left cancellativity as well, then Ore's Theorem gives us embeddability in a group. It seems possible at this point that all left amenable, right cancellative semigroups are in fact cancellative—this is *Sorenson's conjecture*. Sorenson's conjecture is false in general. This problem was resolved by Klawe (1977), who showed that Sorenson's conjecture is equivalent to the Argabright-Wilde conjecture—that every left amenable semigroup satisfies the Strong Følner Condition—and then analysed a semidirect product that produced a useful counterexample.

Sorenson and Klawe both also examined the left measurable semigroups. The result for left measurable semigroups corresponding to Sorenson's conjecture does hold:

Theorem 4.24 If S is a left measurable, right cancellative semigroup, then it is cancellative. (Not shown here—see Klawe, 1977 and Paterson, 1988).

Finally, here is one last theorem of Frey, which closes the circle with respect to subsemigroups of groups. Which subsemigroups of an amenable group are left amenable? Only those that are left reversible.

Theorem 4.25 Let S be a subsemigroup of an amenable group. Then S is left amenable if, and only if, S is left reversible. (Paterson, 1988)

4.4 Amenable inverse semigroups

Inverse semigroups are a broad and interesting class of semigroups. If various definitions do not agree for semigroups in general but do for groups, that may be an indication there is hope for inverse semigroups. However, the next theorem completely characterises amenability of inverse semigroups.

Definition 4.26 The *maximal group homomorphic image* of an inverse semigroup S, denoted $G(S)$ and not to be confused with the group of fractions, is the largest group which can be obtained as the image of a homomorphism.

The *semilattice of idempotents* of an inverse semigroup S, denoted $E(S)$, consists of all idempotent elements. Since the idempotents of an inverse semigroup commute, it is clearly a semilattice.

The *natural partial order* on an inverse semigroup S, denoted $\leq$, is given by setting $a \leq b$ if there exists some $e \in E(S)$ such that $a = eb$.

Lemma 4.27 For any inverse semigroup S, $G(S) \cong S/\rho$, where ρ is defined by setting $a \,\rho\, b$ if and only if there exists some $e \in E(S)$ such that $ea = eb$. Alternatively, $a \,\rho\, b$ if and only if there exists some $c \in S$ such that $c \leq a$ and $c \leq b$. ρ is known as the *minimum group congruence* (Lawson, 1998, p.62), being the smallest congruence that produces a group when factored out. Being the smallest such congruence, it necessarily produces the maximal group image.

A simple consequence is that if $0 \in S$, then $G(S)$ is the trivial group.

Inverse semigroups are either amenable or not amenable (Paterson, 1998, p212), similar to groups. Note also the proof of Lemma 2.2 works exactly the same. The slightly surprising yet anticlimactic theorem due to Duncan and Namioka (1978) is that

Theorem 4.28 An inverse semigroup S is amenable if and only if $G(S)$ is amenable (Duncan and Namioka, 1978; Paterson, 1998).

Proof This proof is adapted from Paterson (1998, Appendix A). Let $\psi : S \to G(S)$ be the maximal group homomorphism.

Suppose S is amenable, and $m \in \ell^\infty(S)^*$ is a left-invariant mean for S. Then define $n \in \ell^\infty(G(S))^*$ by setting

$$n(\phi) := m(\phi \circ \psi) \quad \text{for all } \phi \in \ell^\infty(G(S)).$$

This n is clearly a left-invariant mean for $G(S)$, since for any $g \in G(S)$, there is an $s \in S$ such that $g = \psi(s)$, and then for all $\phi \in \ell^\infty(G(S))$,

$$\begin{aligned} n(g \cdot \phi) &= m(\{g \cdot \phi\} \circ \psi) \\ &= m(x \mapsto \phi(\psi(s)\psi(x))) \\ &= m(x \mapsto \phi(\psi(sx))) \\ &= m(s \cdot (\phi \circ \psi)) \\ &= m(\phi \circ \psi) \\ &= n(\phi). \end{aligned}$$

Conversely, suppose $G(S)$ is amenable. By Theorem 2.11 and Remark 2.12, for every $\epsilon > 0$ and finite $K \subseteq G$ there is a finite mean ν such that $\|\nu - x * \nu\|_1 < \epsilon$ for all $x \in K$. Of course, this holds for semigroups as well. We aim to show that, similarly, for every $\epsilon > 0$ and finite set $H \subseteq S$ there is a finite mean f such that $\|f - y * f\|_1 < \epsilon$ for every $y \in H$.

1. Let $H \subseteq S$ be a finite set and $\epsilon > 0$, and let $K = \psi(H)$. Thus obtain a finite mean $\nu \in \ell^1(G)$ for K and ϵ, and let $F = \mathrm{supp}(\nu)$. (F is finite.)

2. Choose a (finite) $L \subseteq \psi^{-1}(F)$ such that $\psi|_L$ is an injection. Since $t \,\rho\, te$ for any $t \in S$ and $e \in E(S)$, we may vary the choice of L by right-acting by e as necessary.
3. Consider when $e \in E(S)$ is small (such as a zero). Then Le has correspondingly smaller cardinality. Varying over $e \in E(S)$ (in particular, the small idempotents e) it follows that ψ restricted to $HL \cup L$ must be a bijection onto $KF \cup F$.
4. For each $x \in F$ choose a $y_x \in S$ such that $\psi(y_x) = x$. Then let
$$f = \sum_{x \in F} \nu(x)\, y_x.$$
Note that f is a finite mean on S.
5. Since ψ restricted to $HL \cup L$ is a bijection,
$$\|y * f - f\|_1 = \|\psi(y) * \nu - \nu\|_1 < \epsilon \quad \text{for all } y \in H.$$
Thus S is amenable. □

Thus, amenability of inverse semigroups is no more interesting a problem than that of amenable groups and finding the maximal group homomorphism. Since inverse semigroups with zero have maximal group images that are trivial, this result coincides with the previous demonstration that every semigroup with a zero is amenable.

In the literature, this point in particular often leads to a discussion of differing definitions of amenability for inverse semigroups, and how the various definitions of group amenability do not translate well into an inverse semigroup context (e.g. Milan (2007); Paterson (1998)). For the above reasons, amenability as given by Day is regarded as too weak.

An alternative approach on groups is that G is amenable (as a group) if and only if $L^1(G)$ is amenable (as a Banach algebra). Therefore we might consider amenability of the semigroup algebra as a better replacement for amenability of the semigroup itself.

Theorem 4.29 For a discrete inverse semigroup S, $\ell^1(S)$ is amenable (as a Banach algebra) if and only if:

(i) $E(S)$ is finite, and

(ii) every subgroup of S is amenable. (Duncan and Namioka, 1978) □

While this result rules out undesirable examples such as $\mathbb{F}_2^0$ ($\mathbb{F}_2$ is clearly a subgroup), this is possibly too strong to use as a definition, since it many commutative inverse semigroups do not have finite idempotent semilattices. Paterson (1998) suggests the problem is partially resolved using the following theorem regarding $VN(S)$—the von Neumann algebra of S.

Theorem 4.30 Let S be an inverse semigroup, where each maximal subgroup is amenable. Then $VN(S)$ is amenable. (Not shown here.)

4.4.1 Weak containment and inverse semigroups

Milan (2008) argues that the *weak containment property* is an appropriate notion of amenability for inverse semigroups, which is compatible with amenability on the subclass of groups. So, it is worthwhile explaining the weak containment property for inverse semigroups.

Inverse semigroups are naturally thought of as partial bijections of some set, and by the Wagner-Preston theorem *every* inverse semigroup is isomorphic to a subsemigroup of a symmetric inverse monoid. Similarly, consider representations of inverse semigroups using partial isometries of a Hilbert space $\mathcal{H}$. That such representations always exist is a consequence of Wagner-Preston. The *universal C*-algebra of an inverse semigroup* S, denoted $C^*(S)$, is universal for such partial-isometry representations, and is constructed completely analogously to the case for groups (Duncan and Paterson, 1985).

Definition 4.31 Consider a $*$-semigroup S and a (separable) Hilbert space $\mathcal{H}$.

(i) The $*$-semigroup S will be identified with the basis of $\ell^1(S)$, i.e. $s \mapsto \chi_{\{s\}}$, so for all $f \in \ell^1(S)$ we may write

$$f = \sum_{t \in S} f(t)\, t.$$

(ii) The involution on S can be lifted to $\ell^1(S)$ to make $\ell^1(S)$ a Banach $*$-algebra:

$$f^*(s) = \overline{f(s^*)} \quad \text{for all } s \in S.$$

(iii) The C^*-algebra of bounded linear operators on the Hilbert space $\mathcal{H}$ is denoted $\mathcal{B}(\mathcal{H})$. For $T \in \mathcal{B}(\mathcal{H})$ the involution T^* is just the Hilbert space adjoint of T. (Allan, 2011, p.269)

(iv) A *[cyclic] representation on* $\mathcal{H}$ is a $*$-homomorphism to $\mathcal{B}(\mathcal{H})$ [such that the span of the union of the images is norm dense in $\mathcal{H}$].[2]

For example, the $*$-homomorphism $\pi : S \to \mathcal{B}(\mathcal{H})$ is a representation of S on $\mathcal{H}$, and is cyclic if

$$\operatorname{span}\{\pi(s)\,\xi : s \in S, \xi \in \mathcal{H}\}$$

is dense in $\mathcal{H}$.

(v) A representation π of S extends to representations of $\mathbb{C}(S)$ and $\ell^1(S)$ (Barnes, 1976): for all $f \in \ell^1(S)$,

$$\pi(f) := \sum_{t \in S} f(t)\,\pi(t)\,.$$

(vi) The C*-algebra $C^*(S)$ is defined as the completion of $\ell^1(S)$ under the norm[3] given by

$$\|f\| = \sup_{\pi} \|\pi(f)\|$$

where the supremum is taken over all representations π (Paterson, 1998, p.26).[4] $C^*(S)$ is called the *C*-enveloping algebra* of $\ell^1(S)$. A representation of $\ell^1(S)$ on $\mathcal{H}$ extends to a representation of $C^*(S)$ on $\mathcal{H}$.

(vii) The *inverse semigroup of partial isometries of* $\mathcal{H}$ shall be denoted $\mathrm{PI}(\mathcal{H})$. It is clearly a subsemigroup of the inverse semigroup of partial bijections on the underlying set for $\mathcal{H}$. It is also clearly a subalgebra of $\mathcal{B}(\mathcal{H})$.

Lemma 4.32 If S is an inverse semigroup (denoting inverses with $*$, for now), s^*s is an idempotent element for all $s \in S$, and so for a representation $\pi : S \to \mathcal{B}(\mathcal{H})$, we have that $\pi(s^*s) = \pi(s)^*\,\pi(s)$ is a self-adjoint idempotent (a *projection*), and therefore $\pi(s)$ is a partial isometry of $\mathcal{H}$ (Barnes, 1976) and all representations of inverse semigroups on $\mathcal{H}$ are $*$-homomorphisms onto $\mathrm{PI}(\mathcal{H})$.

[2] It is most common to construct representations as any $*$-homomorphisms, and then identify the degenerate representations later. Examples include Duncan and Paterson (1985). But later Paterson (1998) bakes non-degeneracy into his definition of representation by including the denseness in $\mathcal{H}$ as a condition.

[3] As given here, it seems to only be a seminorm, but this is fixed up later.

[4] As given here, $C^*(S)$ depends not only on S but on $\mathcal{H}$ as well, so perhaps it should be denoted $C^*(S, \mathcal{H})$. Usually the $\mathcal{H}$ is obvious from the context.

Lemma 4.33 An important reason to consider $C^*(S)$ is that it is universal for partial isometry representations of S, that is to say, if π is a partial isometry representation of S, and i is the standard inclusion of S as the basis of $C^*(S)$, then there exists a unique $*$-homomorphism $\hat{\pi}$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "\pi"'] & C^*(S) \arrow[d, "\hat{\pi}"] \\ & PI(\mathcal{H}) \end{tikzcd}$$

We call $\hat{\pi}$ the *induced* representation of $C^*(S)$. (Not shown here, but should be reasonably clear from the above.)

Definition 4.34 Let S be inverse and let $\mathcal{H} = \ell^2(S)$. The *left regular representation* $\pi_2 : S \to PI(\ell^2(S))$ is determined by

$$\pi_2(s)(t) = \pi_2(s)(\chi_{\{t\}}) := \begin{cases} \chi_{\{st\}} & \text{if } s^*st = t \\ 0 & \text{otherwise} \end{cases}$$

for all $t \in S$ (Duncan and Paterson, 1985), or more explicitly,

$$\pi_2(s)f := \sum_{tt^* \leq s^*s} f(t)\, st$$

for all $f \in \ell^2(S)$ (Paterson, 1998).

Clearly the natural partial order on S is significant here. It may be interesting to see how the left regular representation might be exented to arbitrary semigroups using the Mitsch order as the generalisation of the regular order. The necessity of limiting the summation to idempotents below s^*s is explicit if we write the left regular representation as

$$\{\pi_2(s)(f)\}(u) = \sum_{\substack{t \in s^{-1}\{u\} \\ tt^* \leq s^*s}} f(t)\,.$$

For example, if $0 \in S$, since $tt^* = 0$ is the only idempotent satisfying $tt^* \leq 0$, we have $\{\pi_2(0)(f)\}(0) = f(0)$. In the case of a group, the only idempotent is the identity element, but $g^{-1}\{h\} = \left\{g^{-1}h\right\}$ and so $\{\pi_2(g)(f)\}(h) = f\left(g^{-1}h\right)$.

The $*$-homomorphism induced by the left regular representation is $\widehat{\pi_2} : C^*(S) \to \mathcal{B}(\ell^2(S))$.

Definition 4.35 The image of $\widehat{\pi_2}$ is $C^*_r(S)$, known as the *reduced C*-algebra of* S. Equivalently, $C^*_r(S)$ is the norm closure of $\pi_2(\ell^1(S))$.

Therefore we can consider the extension of π_2 to $\widehat{\pi_2} : C^*(S) \to C^*_r(S)$ as a surjective $*$-homomorphism. As noted by Duncan and Paterson (1985), while π_2 is faithful (on $\ell^1(S)$), $\widehat{\pi_2}$ need not be. Parallel to the group case, there is the following.

Definition 4.36 An inverse semigroup S has the *weak containment property* if and only if each irreducible partial isometry representation π is weakly contained in the left regular representation π_2.

Definition/Theorem 4.37 An inverse semigroup S has the weak containment property if, and only if, $\widehat{\pi_2}$ is an isomorphism—alternatively, this is how the weak containment property is defined. (Duncan and Paterson, 1985; Milan, 2008).

Theorem 4.38 A group G is amenable if and only if it has the weak containment property. (Pavel, 2007)

Example 4.39 Every semilattice has the weak containment property. (Milan, 2008)

Example 4.40 The free group on two generators with zero, $\mathbb{F}_2^0$, does not have the weak containment property. On the other hand, all commutative inverse semigroups have the weak containment property. (Milan, 2008)

Definition/Theorem 4.41 Suppose $\phi : S \to G$ is a homomorphism from the inverse semigroup S onto some group G, with $\ker(\phi) = H$. If $\|x\|_{C^*(S)} = \|x\|_{C^*(H)}$ for all $x \in \mathbb{C}(H)$, then H is said to be *C*-isometric in* S.

If H is C*-isometric in S then S has the weak containment property if, and only if, G is amenable and H has the weak containment property. (Milan, 2008)

To set up the next example, a couple of definitions are in order.

Definition 4.42 An inverse semigoup S is E-*unitary* if, for all $e \in E(S)$ and $s \in S$, $es \in E(S) \Rightarrow s \in E(S)$—that is, wherever e is an idempotent and $e \leq s$, then s is also an idempotent. (Lawson, 1998, p57)

A congruence σ on an inverse semigroup S is *idempotent pure* if, for all $a \in S, e \in E(S)$ such that $a\,\sigma\,e$, then a is an idempotent—in other words, $E(S)$ is a σ-equivalence class. (Lawson, 1998, p65)

These two definitions are bundled together here because every E-unitary inverse semigroup S has an idempotent pure minimum group congruence, and vice-versa (Lawson, 1998, p66). Clearly if S is an inverse semigroup with zero and is E-unitary, then it can only be a semilattice, since 0 is an idempotent less than every element in the natural partial order. This is again solved by excluding 0 in a particular way. An inverse semigroup with zero S is said to be E^*-*unitary* if, for every $e \in E(S)\setminus\{0\}$ and $s \in S$, $e \leq s \Rightarrow s \in E(S)$ (Lawson, 1998, Ch. 9)

Example 4.43 Suppose S is an E-unitary inverse semigroup. Then the minimum group congruence of S is idempotent pure, and it follows that the kernel of the maximum group homomorphism is the exactly E, a semilattice. Every semilattice has the weak containment property, and it is possible to show that E is C^*-isometric in S. Thus as a corollary to Theorem 4.41, S has weak containment if and only if $G(S)$ is amenable. (Milan, 2008)

Example 4.44 Graph inverse semigroups have the weak containment property (Milan, 2008). This class of inverse semigroups includes and generalises the polycyclic monoids on n generators P_n (Jones and Lawson, 2011). P_n is the graph inverse semigroup associated with the n-rose (bouquet of n circles).

A definition of the graph inverse semigroups will be given in the next chapter.

Chapter 5

Fairly Amenable Semigroups

The first three sections of this chapter, in a condensed form, have been sent in the form of an article, to the Semigroup Forum for potential publication.

5.1 Definitions

Recall the left and right regular representations λ and ρ from Definition 1.2. These formalise thinking of multiplying on the left and multiplying on the right by an element as self-maps of a semigroup S.

If $\lambda_s|_A$, that is, λ_s on the restricted domain of A, is a bijection $A \to sA$, then $|A| = |sA|$. Since λ_s necessarily maps onto sA, bijectivity of $\lambda_s|_A$ depends only on injectivity. This motivates the following definition.

Definition 5.1 If $\lambda_s|_A : A \to sA$ is an injection, then s is said to *act injectively on the left of* A. If $\rho_s|_A : A \to As$ is an injection, then s acts injectively on the *right* of A.

Note that all group elements act injectively on all subsets of the group, since the self-action consists entirely of bijections. A trivial but important example of a non-injective action of a semigroup is the action of a zero element. A non-trivial example might be a semigroup describing chess moves (for an application of semigroups in chess see Morse and Hedlund, 1944). In the game of chess, pawns cannot move backwards and chess pieces are generally non-renewable, so the set of legal moves is gradually reduced during a game, and there are multiple sequences of moves that produce the same board.

There are a number of obvious statements to make about sets on which an element acts injectively.

Lemma 5.2 For any $s \in S$, let $LI_s \subseteq \mathcal{P}(S)$ be the collection of sets on which s acts injectively on the left. The following all hold.

(i) If $A \in LI_s$, and $B \subseteq A$, then $B \in LI_s$. Thus the collection LI_s is always *downwards-closed*. It is then trivial that for any sets $A, B \in LI_s$ that $A \cap B \in LI_s$, i.e. the finite intersection property.

(ii) For any $s \in S$, $S \in LI_s$ if, and only if, $LI_s = \mathcal{P}(S)$.

(iii) $LI_s = \mathcal{P}(S)$ for all $s \in S$ if, and only if, S is left cancellative.

(iv) $\emptyset \in LI_s$, and also $\{x\} \in LI_s$ for all $x \in S$. It is possible that there are no other sets in LI_s, see for example LI_0.

(v) If $A \in LI_s$ and there is no proper superset $B \supset A$ such that $B \in LI_s$, then call A *maximal* in LI_s. There is always at least one maximal set in LI_s, since it cannot be empty and it is bounded above by S. If $A \in LI_s$ is maximal then there exists a maximal $B \subseteq S \backslash A$.

(vi) By induction on item (v) above, one or more collections of maximal sets from each LI_s form a partition of S, possibly an infinite partition.

(vii) Since non-cancellativity of s on the left of any set A is evident from some two-element subset of A, for every $s \in S$ either $LI_s = \mathcal{P}(S)$ or there is some two-element set F such that $F \notin LI_s$.

The last point is embellished further here.

Lemma 5.3 For any $s \in S$ and $A \subseteq S$, the following are equivalent.

(i) s acts injectively on the left of A.

(ii) For all two-element sets $F \subseteq A$, $|sF| = |F|$.

(iii) For any finite set $F \subseteq S$, $|s(F \cap A)| = |F \cap A|$.

Finally, there is a neat trick that will be used a few times throughout the remainder of this work.

Lemma 5.4 Let S be a semigroup. For each $s \in S$ let θ_s be the equivalence relation on S defined by setting $x \,\theta_s\, y \Leftrightarrow sx = sy$ for all $x, y \in S$. Every non-empty set of the form $s^{-1}\{x\}$ corresponds to a θ_s-equivalence class.

Proof Observe that for all x, if $s^{-1}\{x\}$ is non-empty then there is a $y \in S$ such that $sy = x$, and then

$$\begin{aligned} s^{-1}\{x\} &= \{t \in S : st = x\} \quad \text{by definition} \\ &= \{t \in S : st = sy\} \\ &= y\theta_s, \end{aligned}$$

as required. □

This is a particularly useful way to think about injectivity, because s acts injectively on the left of a set A if, and only if,

$$\left|A \cap s^{-1}\{x\}\right| \leq 1 \quad \text{or, equivalently,} \quad |A \cap x\theta_s| \leq 1$$

for all $x \in S$.

Definition 5.5 (Subinvariant) Let S be a semigroup, and μ a finitely-additive measure on S with finite total measure. If

$$\mu(sA) \leq \mu(A) \quad [\mu(As) \leq \mu(A)] \quad \text{for all } s \in S \text{ and } A \subseteq S,$$

then we say μ is *left [right] sub-invariant*.

Suppose $sA = A$, for instance s is an identity. It is then clear that the inequality above cannot be strict in general.

Suppose for some element s and finite set A there is some s' such that $s'sA = A$. Then both s and s' act injectively, and when restricted to A, $s's$ acts *bijectively*—a permutation of A. Furthermore, if μ is left sub-invariant,

$$\mu(A) = \mu(s'sA) \leq \mu(sA) \leq \mu(A),$$

and thus $\mu(sA) = \mu(A)$. This suggests the next definition, which is the most important here.

Definition 5.6 (Fairly invariant, fairly amenable) Let S be any semigroup, let μ a finitely-additive measure on S with $\mu(S) = 1$, and let $s \in S$ and $A \subseteq S$.

If whenever s acts injectively on the left [right] of A,

$$\mu(sA) = \mu(A) \quad [\mu(As) = \mu(A)]$$

then μ is *fairly left [right] invariant*. If such a μ exists for a given semigroup S, then S is *fairly left [right] amenable*.

In other words, invariance of μ is only required in the places where an element s acts injectively on the set. As we shall see, this weakening of total invariance handles the issue discussed in van Douwen (1992, p231).

Lemma 5.7 For any semigroup S and finitely-additive probability measure μ, left [right] fair invariance of μ implies left [right] sub-invariance of μ.

Proof For a pictoral overview see Figure 5.1.

1. For any $A \subseteq S$ and $s \in S$ there exists a $B \subseteq A$ such that $sA = sB$ and s is injective on B.
   PROOF: Use the Axiom of Choice to choose one $b \in s^{-1}\{x\} \cap A$ for each $x \in sA$. B is simply the set of those choices.
2. If $B \subseteq A \subseteq S$, and $sA = sB$, and s acts injectively on B (but not necessarily on A), then $\mu(A) \geq \mu(sA)$.
   PROOF:
$$\begin{aligned} \mu(A) &\geq \mu(B) && \because B \subseteq A \\ &= \mu(sB) && \because \text{fair invariance of } \mu \\ &= \mu(sA) && \because sB = sA. \end{aligned}$$
   as required. □

Remark 5.8 What about selecting $\mu(sA) \geq \mu(A)$ as a condition ("super-invariance")? If sA is a subset of A then $\mu(sA) = \mu(A)$, and so disjoint subsets sA, tA may lead to a contradiction.

By definition, if s acts injectively on the left of A, then $sa = sb \Rightarrow a = b$ for any $a, b \in A$ and s is *left cancellative on* A, similarly on the right. Hence another way of defining fair invariance is in terms of cancellation. Groups are totally cancellative both ways, but there are non-group examples of left- and right-cancellative semigroups.

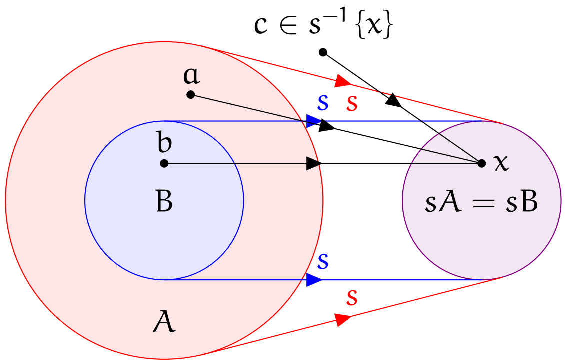

Figure 5.1: For every set A and element s there is some subset B such that $sA = sB$ and s acts injectively on B.

5.2 Consequences

5.2.1 Groups and Følner criteria

Fair amenability is a generalisation of amenability for groups, as follows.

Corollary 5.9 A group is amenable if, and only if, it is fairly amenable.

Proof This is trivial since every element g in a group G acts bijectively on G, and so a finitely-additive measure on G is invariant if, and only if, it is fairly invariant. □

Similar to classical amenability, fair amenability is also a consequence of a Følner-type condition, as follows.

Theorem 5.10 Let S is a countable semigroup. If there exists a sequence of non-empty finite sets $\{F_n\}_{n\in\mathbb{N}}$ eventually covering S such that

$$\lim_{n\to\infty} \frac{|sF_n \triangle (sS \cap F_n)|}{|F_n|} = 0 \quad \text{for all } s \in S,$$

then S is left fairly amenable. (Similarly for $F_n s$, Ss,on the right.)

Proof Fix a free ultrafilter $\mathcal{U}$ over $\mathbb{N}$ and define μ through the ultralimit

$$\mu(A) := \lim_{\mathcal{U}} \frac{|A \cap F_n|}{|F_n|} \quad \text{for all } A \subseteq S.$$

1. For any set A, the ultralimit above exists, and μ is a finitely-additive measure with $\mu(S) = 1$.

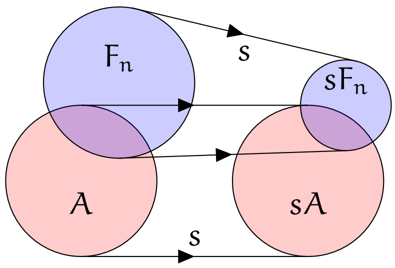

Figure 5.2: If s acts injectively on A, then it also does so on $A \cap F_n$, and so $|A \cap F_n| = |sA \cap sF_n|$.

PROOF: This is shown in a manner identical to Lemma 2.15.

2. μ is left fairly invariant.

   PROOF: Suppose s acts injectively on the left of A. Then $|A \cap F_n| = |s(A \cap F_n)| = |sA \cap sF_n|$ (see Figure 5.2), and so

$$\begin{aligned}
\left| \frac{|A \cap F_n|}{|F_n|} - \frac{|sA \cap F_n|}{|F_n|} \right| &= \frac{||A \cap F_n| - |sA \cap F_n||}{|F_n|} \\
&= \frac{||sA \cap sF_n| - |sA \cap F_n||}{|F_n|} \\
&\leq \frac{|sA \cap (sF_n \triangle F_n)|}{|F_n|} \\
&\leq \frac{|sS \cap (sF_n \triangle F_n)|}{|F_n|} \\
&= \frac{|sF_n \triangle (sS \cap F_n)|}{|F_n|} \to 0 \text{ as } n \to \infty
\end{aligned}$$

   by hypothesis, and hence $\mu(A) = \mu(sA)$, as required. □

Remark 5.11 Since $2|F_n \backslash sF_n| \geq |sF_n \triangle F_n| \geq |F_n \backslash sF_n|$ for any $s \in S$, one may as usual substitute $F_n \backslash sF_n$ for $sF_n \triangle F_n$ in the Følner condition. Note that while there are semigroups lacking strong Følner sequences that are also fairly amenable, this appears to be mitigated by the intersection with the right ideal sS. Consider, for example, an infinite amenable group G with zero adjoined (G^0), which is fairly amenable (see Corollary 5.20 below) and the zero element has no associated strong Følner sequence, however $0S = \{0\}$ and therefore any Følner sequence will do with respect to 0.

5.2.2 Basic consequences

Corollary 5.12 All finite semigroups S are fairly amenable (both ways).

Proof The element s is injective on the left of $A \subseteq S$ if, and only if, $|sA| = |A|$, similarly on the right. Therefore the counting measure suffices. Alternatively, use the constant Følner sequence $\{S\}_{n\in\mathbb{N}}$. □

Remark 5.13 Suppose that, given some set A, $\mu(sA) = \mu(A)$ $[\mu(As) = \mu(A)]$ for any s. We may describe A as being a left [right] μ-invariant set. In a fairly left [right] amenable semigroup S, every singleton set $\{x\}$ for $x \in S$ is guaranteed to be a left [right] invariant set.

Lemma 5.14 Let S be a semigroup that is both infinite and left [right] fairly amenable with measure μ, having a left [right] zero $z \in S$. If F is a finite subset of S, then $\mu(F) = 0$. Such a μ is said to be *diffuse*, as defined by van Douwen (1992, p225).

Proof

1. Every singleton set has the same measure k.
   Proof: We can go via $\{z\}$: for any $s, t \in S$,
   $$\mu(\{s\}) = \mu(z\{s\}) = \mu(\{z\}) = \mu(z\{t\}) = \mu(\{t\}).$$
2. $k = 0$, therefore $\mu(F) = 0$.
   Proof: If $k > 0$ there exists some finite N such that $Nk > 1$, i.e. the disjoint union of N singletons would have greater than 1 measure. Hence $k = 0$. Then
   $$\mu(F) = \sum_{f\in F} \mu(\{f\}) = \sum_{f\in F} k = 0.$$
   The right case holds similarly. □

Corollary 5.15 Let S be a non-trivial semigroup with zero. The finitely-additive measure δ_0 given by

$$\delta_0(A) = \delta_0\left(0^{-1}A\right) = \delta_0\left(A0^{-1}\right) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{if } 0 \notin A \end{cases}$$

(i.e. the measure obtained from the invariant mean $m \in \ell^\infty(S)^*$ given by $m(f) = f(0)$ for all $f \in \ell^\infty(S)$) cannot be fairly invariant.

Proof

$$\begin{aligned} 1 &= \delta_0(\{0\}) && \text{by definition} \\ &= 0 && \because \text{Lemma 5.14,} \end{aligned}$$

contradiction. □

Corollary 5.16 If a semigroup S is left measurable (see Definition 4.1), then it is left fairly amenable.

Proof We are given a totally invariant finitely-additive measure. This condition includes all cases, in particular where the element acts injectively on the left. □

Cancellative semigroups

Corollary 5.17 Suppose S is a left cancellative semigroup. Clearly every element acts injectively on the left. Therefore S is left fairly amenable if, and only if, S is left measurable. □

Hence we may directly import the investigations into left measurable semigroups of Sorenson and of Klawe (1977).

Corollary 5.18 The following hold for a semigroup S.

(i) If S is left cancellative and left fairly amenable, then S is left measurable, and thus S is left reversible.

(ii) If S is cancellative and left fairly amenable, then it is embeddable in a group.

(iii) Suppose S is right cancellative. Then S is left measurable if, and only if, S is left fairly amenable and left cancellative.

Proof

(i) Suppose S is left cancellative and left fairly amenable with measure μ. Every action on the left of a set is injective, therefore μ is a left measure. By a similar proof to that of Theorem 4.20, it is left reversible.

(ii) By the above point, we have left reversibility in addition to cancellativity, and therefore it is embeddable.

(iii) Sorenson's theorem (Theorem 4.24) is that if S is left measurable and right cancellative, then it is also left cancellative. If it is left measurable then it is also left fairly amenable. The converse is given above. □

A left fairly amenable semigroup that is not left cancellative may not necessarily be left reversible. If S is not left cancellative, then there is at least one element $a \in S$ such that a does not act injectively on the left of S, thus aS may have measure strictly less than one, and so the proof of Theorem 4.20 falls down here.

5.2.3 Subsemigroups

Every subgroup of an amenable group is amenable, including those subgroups having measure zero. A quick summary of this proof goes as follows: let G be an amenable group with measure μ, H a subgroup. Choose a set M of representatives from each left coset of H, then define a measure ν on H by setting $\nu(A) := \mu(MA)$ for all $A \subseteq H$ (Wagon, 1993, p149). It would be nice to emulate this in the semigroup case, but it seems there is no adequate analogue for semigroups of the coset structure of a group. Perhaps the obvious should be stated:

Lemma 5.19 Let S be a left [right] fairly amenable semigroup with measure μ, and let T be a subsemigroup of S having $\mu(T) > 0$. T is then left [right] fairly amenable.

Proof We may use ν as given by $\nu(A) = \mu(A)/\mu(T)$ for all $A \subseteq T$. □

This mirrors the classical case (Day, 1957, p.518). In particular, any subgroup G of a left or right fairly amenable semigroup is amenable *provided that* $\mu(G) > 0$.

Corollary 5.20 Let S be a semigroup without zero. S^0 is left [right] fairly amenable if and only if S is. In particular, if G is a group, G^0 is fairly amenable if and only if G is amenable.

Proof Since the finite case is trivial, assume that S is infinite. If S^0 is left fairly amenable with μ', since S^0 contains a zero, by Lemma 5.14 $\mu'(\{0\}) = 0$, which by finite additivity implies $\mu'(S) = 1$. By Lemma 5.19 S is fairly amenable and, in the case of a group, amenable by Corollary 5.9.

Conversely, if S is left fairly amenable with some μ then assigning $\mu'(A) = \mu(A \cap S)$ yields a fairly invariant measure μ' on S^0. The case on the right holds similarly. □

5.2.4 Green's relations

Definition 5.21 *Green's relations* $\mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{H}$, and $\mathcal{J}$ on a semigroup S are defined by the following, for all $x, y \in S$. (Howie, 1976)

($\mathcal{L}$) $x \mathcal{L} y$ if there exists $a, b \in S^1$ such that $ax = y$ and $by = x$. Alternatively, $x \mathcal{L} y \Leftrightarrow S^1x = S^1y$ (x and y generate the same principal left ideal).

($\mathcal{R}$) Similarly, $x \mathcal{R} y$ if there exists $a, b \in S^1$ such that $xa = y$ and $yb = x$. Alternatively, $x \mathcal{R} y \Leftrightarrow xS^1 = yS^1$.

($\mathcal{J}$) $x \mathcal{J} y$ if there exists $a, b, c, d \in S^1$ such that $axb = y$ and $cyd = x$. Alternatively, $x \mathcal{J} y \Leftrightarrow S^1xS^1 = S^1yS^1$.

($\mathcal{D}$) $\mathcal{D} := \mathcal{L} \vee \mathcal{R}$. It so happens that $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.

($\mathcal{H}$) Finally, $\mathcal{H} := \mathcal{L} \cap \mathcal{R}$. Every $\mathcal{H}$-class contains at most one idempotent, and those that do are subgroups of S.

It is also usual to induce, from the inclusion of the principal ideals, orderings of the $\mathcal{L}$-classes, the $\mathcal{R}$-classes, and the $\mathcal{J}$-classes, that is, if L_a, L_b are the $\mathcal{L}$-classes containing a, b (similarly, R_a, R_b and J_a, J_b) then

- $L_a \leq L_b \Leftrightarrow S^1a \subseteq S^1b$,
- $R_a \leq R_b \Leftrightarrow aS^1 \subseteq bS^1$, and
- $J_a \leq J_b \Leftrightarrow S^1aS^1 \subseteq S^1bS^1$.

If every set of $\mathcal{L}$-classes contains a minimal $\mathcal{L}$-class in the ordering, then S is said to satisfy the condition $\min_L$, and similarly for $\mathcal{R}$-classes and $\min_R$, and $\mathcal{J}$-classes and $\min_J$.

Lemma 5.22 If S satisfies $\min_L$ and $\min_R$, then $\mathcal{D} = \mathcal{J}$. (Howie, 1976, p41)

In the present study, there are two easy lemmas that take advantage of Green's relations. The first describes sets for any semigroup on which a fairly-invariant measure must be diffuse.

Lemma 5.23 If S is left [right] fairly amenable with measure μ, any finite subset F of an infinite $\mathcal{L}$-class [$\mathcal{R}$-class] has $\mu(F) = 0$. It follows that in either case any finite subset F of an $\mathcal{H}$-class has $\mu(F) = 0$, and if S is fairly amenable on both sides than any finite subset F of a $\mathcal{D}$-class has $\mu(F) = 0$.

Proof

1. Every singleton subset of an $\mathcal{L}$-class has the same measure k.
   PROOF: By definition, for all $a, b \in S$ such that $a \mathcal{L} b$, there exists $s, s' \in S^1$ such that $sa = b, s'b = a$, and we only need one of these to establish that if μ is the left fairly invariant finitely-additive measure,
   $$\mu(\{a\}) = \mu(s\{a\}) = \mu(\{sa\}) = \mu(\{b\}) \quad \text{for all } a, b \in S.$$
2. Every finite subset has measure 0.

PROOF: As for the final step of Lemma 5.14. □

Green's Lemma (Howie, 1976, p43) states that for any $a, b \in S$ such that $a \mathcal{R} b$, the restricted right regular representations to $\mathcal{L}$-classes, $\rho_s|L_a$ and $\rho_{s'}|L_b$, are mutually inverse $\mathcal{R}$-class preserving bijections between the $\mathcal{L}$-classes L_a and L_b. Put another way, there exists an $s \in S$ that acts injectively on the right of L_a and an $s' \in S$ that acts injectively on the right of L_b.

Lemma 5.24 Let S be a semigroup.

If S is right fairly amenable with measure μ then within each $\mathcal{D}$-class all $\mathcal{L}$-classes have equal μ measure, and furthermore, every $\mathcal{H}$-class in the same $\mathcal{R}$-class has the same μ measure across the $\mathcal{L}$-classes.

Similarly, if S is left fairly amenable with μ then within each $\mathcal{D}$-class all $\mathcal{R}$ classes have equal μ measure, and furthermore, every $\mathcal{H}$-class in the same $\mathcal{L}$-class has the same μ measure across the $\mathcal{R}$-classes.

It follows that if S is fairly amenable (both ways) then all $\mathcal{D}$-related $\mathcal{H}$-classes have equal μ measure.

Proof Suppose L_a, L_b are $\mathcal{L}$-classes contained within the same $\mathcal{D}$-class, and let $H_a \subseteq L_a$, $H_b \subseteq L_b$ be $\mathcal{H}$-classes sharing the same $\mathcal{R}$-class.

1. There exist $s, s' \in S^1$ such that $L_a = L_b s'$ and $L_b = L_a s$ are both examples of s and s' acting injectively on the right.

   PROOF: Use Green's Lemma.
2. $\mu(L_a) = \mu(L_a s) = \mu(L_b)$.
3. The right multiplication of H_a by s and H_b by s' both preserve the $\mathcal{R}$-class, and therefore $H_a s = H_b$ and $H_b s' = H_a$.

   PROOF: Green's Lemma again.
4. $\mu(H_a) = \mu(H_a s) = \mu(H_b)$.

The other side holds analogously. □

What can we say about the value of a fairly invariant finitely-additive measure μ between distinct $\mathcal{D}$-classes? Probably not a lot (see Example 5.44 below).

5.2.5 Direct products

A result for groups states that the direct product of finitely many amenable groups is also amenable. This is easily shown by noting that if $G = G_1 \times G_2$ then the subgroup

$H = \{(g_1, 1_{G_2}) : g_1 \in G_1\} \cong G_1$, and $G/H \cong G_2$, so therefore the amenability of G_1 and G_2 imply the amenability of H and G/H, and hence G. The fair amenability analogue of this result is as follows, but shown by explicitly constructing a finitely-additive measure.

Theorem 5.25 Let S, T be semigroups that are each left [right] fairly amenable. Then $S \times T$ is as well.

Proof Let μ_S and μ_T witness the left fair amenability of S and T respectively. Let π_S, π_T denote the projections from $\mathcal{P}(S \times T)$ onto $\mathcal{P}(S)$ and $\mathcal{P}(T)$, respectively.

1. Define μ, on $S \times T$, for each rectangle $R = A \times B$ where $A \subseteq S$ and $B \subseteq T$:
$$\mu(R) := \mu_S(\pi_S(R))\, \mu_T(\pi_T(R)) = \mu_S(A)\, \mu_T(B),$$
which, while not yet defined for all subsets of $S \times T$, is clearly left fairly invariant and finitely-additive, and with $\mu(S \times T) = \mu_S(S)\, \mu_T(T) = 1$.
2. It follows that
$$\mu\left(\bigcup_{i\in I} R_i\right) = \sum_{i\in I} \mu(R_i),$$
for each finite collection of disjoint rectangles[1] $\{R_i\}_{i\in I}$, and this is also left fairly invariant.
Proof: If (s, t) acts injectively on $\bigcup_{i\in I} R_i$, then s acts on $\pi_S(R_i)$ injectively for each $i \in I$, likewise for $t \in T$ on $\pi_T(R_i)$. Furthermore, (s, t) preserves the disjointness of $\{R_i\}_{i\in I}$.
3. Let C be an arbitrary subset of $S \times T$. C is not necessarily a rectangle, so extend μ using
$$\mu(C) := \sup \mu\left(\bigcup_{i\in I} R_i\right),$$
where the supremum is taken over all finite collections of subrectangles of C.
4. μ is then defined for all subsets C of $S \times T$, and is left fairly invariant.
Proof: If $(s, t) \in S \times T$ acts injectively on C then it acts injectively on any finite collection of disjoint subrectangles of C. Each finite collection of disjoint subrectangles of $(s, t)C$ has the form $\{(s, t)R_i\}_{i\in I}$ for a finite collection of disjoint

[1] Take care to avoid confusing *finite collections of rectangles* with *collections of finite rectangles*.

subrectangles $\{R_i\}_{i\in I}$ of C. Hence

$$\begin{aligned}\mu((s,t)C) &= \sup \mu\left(\bigcup_{i\in I}(s,t)R_i\right)\\ &= \sup \sum_{i\in I} \mu((s,t)R_i)\\ &= \sup \sum_{i\in I} \mu(R_i)\\ &= \sup \mu\left(\bigcup_{i\in I} R_i\right)\\ &= \mu(C),\end{aligned}$$

as required. □

For some examples, consider some bands. Recall that, in the classical theory, a right zero semigroup is left amenable but not right amenable.

Example 5.26 Let S be a left (or right) zero semigroup. S is fairly amenable (both sides).

Proof The finite case is handled by Corollary 5.12, so assume S is an infinite left zero semigroup.

1. Any finitely-additive measure μ with $\mu(S) = 1$ is right fairly invariant.
   PROOF: For any $A \subseteq S$ and $s \in S$, $As = A$, so $\mu(As) = \mu(A)$ trivially.
2. There are infinitely many finitely-additive measures μ with $\mu(S) = 1$ that are left fairly invariant.
   PROOF: For any $A \subseteq S$ and $s \in S$, $sA = \{s\}$, and by Lemma 5.14 every $\mu(\{s\}) = 0$ if μ is fairly invariant, but since singletons are the only sets injectively acted on on the left, the following suffices. Fix any free ultrafilter U, and define $\mu(A) = \chi_U(A)$.

The argument holds on the right analogously. □

Example 5.27 Every rectangular band is fairly amenable.

Proof We have just seen the specific examples of left and right zero semigroups (Example 5.26). Each rectangular band is isomorphic to the product of a left zero semigroup and a right zero semigroup, therefore by Theorem 5.25 all rectangular bands are fairly amenable. □

5.2.6 Quotients of semigroups

Another theorem on groups states that if a group G is amenable and $N \lhd G$, then G/N is also amenable. Since every congruence on a group arises as the cosets of a normal subgroup, this means that every quotient of an ameable group is amenable. Given an amenable G with measure μ, we may set ν on G/N using

$$\nu(A) = \mu\left(\bigcup A\right) \quad \text{for all } A \subseteq G/N. \tag{5.1}$$

The corresponding situation in fairly amenable semigroups encounters problems. Let σ be a congruence on a fairly left amenable semigroup S with measure μ. Clearly ν has total measure 1 and is finitely-additive. However it is not always going to be left fairly invariant.

Example 5.28 As described in Proposition 5.48, the free Abelian semigroup on two generators S is fairly amenable with the measure μ. Let σ be the congruence on S with $(b, b^2), (b, ab) \in \sigma$, i.e.

$$S/\sigma \cong \text{sgp}\left\langle a, b \middle| ab = ba = b^2 = b\right\rangle.$$

Now, S/σ is fairly amenable as it is a free commutative semigroup on one generator with a zero, however ν as in Equation 5.1 is not fairly invariant since $\nu(A) = \nu\left((b\sigma)^{-1}A\right)$ (the Dirac delta measure), via Lemma 5.23. □

So, what extra conditions on the congruence permits transferring fair amenability from a semigroup to its quotient? Here is one.

Definition 5.29 Let S be a semigroup and σ a congruence on S such that whenever $s\sigma$ acts injectively on $A \subseteq S/\sigma$, then there exists some $s' \in s\sigma$ which acts injectively on $\bigcup A$ (and therefore on each $a\sigma \in A$), with $s' \bigcup A = \bigcup(s\sigma)A$. Then σ is said to be *left convenient*[2].

Lemma 5.30 If S is a left [right] fairly amenable semigroup with μ, and σ is a congruence on S which is left [right] convenient, then S/σ is also left [right] fairly amenable.

Proof Let ν be a finitely-additive measure defined by setting

$$\nu(A) := \mu\left(\bigcup A\right) \quad \text{for all } A \subseteq S/\sigma.$$

[2]"Convenient" for the purposes of proving the theorem. A better, more descriptive, and enduring name is needed, but not yet forthcoming.

Then

$$\begin{aligned}\nu((s\sigma)A) &= \mu\left(\bigcup(s\sigma)A\right)\\ &= \mu\left(s'\bigcup A\right)\\ &= \mu\left(\bigcup A\right)\\ &= \nu(A)\end{aligned}$$

as required. □

5.2.7 Completely 0-simple semigroups

Completely simple and completely 0-simple semigroups are good classes of semigroups on which to try some of the previous ideas.

Definition 5.31 A *0-simple semigroup* is a semigroup with zero S in which S has no two-sided ideals other than S and $\{0\}$, and $S^2 \neq \{0\}$. (Howie, 1976)

This is equivalent to the condition where the only $\mathcal{J}$-classes of S are $S\setminus\{0\}$ and $\{0\}$. Theorems for *simple* semigroups, i.e. those with no proper two-sided ideals at all, are very often a special case of the theorems for 0-simple semigroups, so focus is on the 0-simple case. Two-sided ideals are of interest when defining simplicity, as opposed to left and right ideals, because any semigroup that is both left and right simple is a group.

Definition 5.32 A *completely 0-simple* semigroup is 0-simple and satisfies the conditions $\min_L$ and $\min_R$. (Howie, 1976)

Some basic facts about completely 0-simple semigroups are as follows.

- Every completely 0-simple semigroup is regular.
- By Lemma 5.22, $\mathcal{D} = \mathcal{J}$, so every completely 0-simple semigroup is also *0-bisimple*, i.e. the only $\mathcal{D}$ classes are $S\setminus\{0\}$ and $\{0\}$.
- Every $\mathcal{H}$-class H of a completely 0-simple semigroup is either a group or $H^2 = \{0\}$.

- Every non-zero idempotent in a completely 0-simple semigroup is *primitive*. A primitive idempotent is one that is minimal but non-zero in the natural partial order. A 0-simple semigroup is completely 0-simple if it contains a primitive idempotent.

A remarkable construction of Rees lets us understand the structure of *all* completely 0-simple semigroups.

Definition 5.33 A *regular* matrix P has at least one non-zero entry in each row and column. The *Rees matrix semigroup* $\mathcal{M}^0(G; I, \Lambda; P)$, where G is a group, I and Λ are index sets, and P is a $\Lambda \times I$ regular "sandwich" matrix with entries in G^0, is defined by using as the set of elements

$$S = (I \times G \times \Lambda) \cup \{0\}$$

and defining the product by

$$(i, a, \lambda)(j, b, \mu) = (i, a p_{\lambda j} b, \mu) \quad \text{for all } i, j \in I; a, b \in G; \lambda, \mu \in \Lambda,$$

where $p_{\lambda j}$ denoting the element of P in position λ, j. If $p_{\lambda j} = 0$, or either side is 0, then the product is 0 also. The regular property of P guarantees the construction is 0-simple.

Theorem 5.34 (Rees's Theorem) Every completely 0-simple semigroup is isomorphic to $\mathcal{M}^0(G; I, \Lambda; P)$ for some group G, index sets I and Λ, and regular sandwich matrix P. Conversely, every such Rees matrix semigroup is completely 0-simple. (Howie, 1976)

In the case that no element of P is 0, then 0 has no proper divisors and is removable, i.e. the semigroup is merely a completely simple semigroup with a zero adjoined. We can define the analogous Rees matrix semigroup for completely simple semigroups, denoted $\mathcal{M}(G; I, \Lambda; P)$—note the mising superscript 0—to be the semigroup with the same multiplication given above, but on the set $I \times G \times \Lambda$ only. The analogous theorem to Rees's Theorem states that every completely simple semigroup is isomorphic to some $\mathcal{M}(G; I, \Lambda; P)$, and the construction is always completely simple.

It pays for us to understand what makes $\mathcal{M}^0(G; I, \Lambda; P)$ 0-simple. Let (i, a, λ) and (j, b, μ) be non-zero elements. Since P is regular, there exist $k \in I$ and $\nu \in \Lambda$

such that $p_{\nu i} \neq 0 \neq p_{\lambda k}$. Then

$$\begin{aligned}(j, p_{\nu i}^{-1}, \nu)(i, a, \lambda)(k, p_{\lambda k}^{-1}, \mu) &= (j, p_{\nu i}^{-1} p_{\nu i} a p_{\lambda k} p_{\lambda k}^{-1}, \mu) \\ &= (j, b, \mu),\end{aligned}$$

hence we can go from any element to any other by multiplying on the left and the right.

Some semigroups considered in previous sections turn out to be completely simple or completely 0-simple semigroups, which can be seen by picking G, I, Λ and P appropriately.

- If $I = \{1\} = \Lambda$, then $\mathcal{M}(G; I, \Lambda; P)$ is isomorphic to G. The $\mathcal{M}^0$ case is isomorphic to the group with zero adjoined.
- If G is the trivial group then $\mathcal{M}(G; I, \Lambda; P)$ is the rectangular band $I \times \Lambda$.
- If P is completely trivial, i.e. P is a matrix with all entries equal to 1_G, then $\mathcal{M}(G; I, \Lambda; P)$ is the direct product $I \times G \times \Lambda$, i.e. the product of G and the rectangular band $I \times \Lambda$. This is called a *rectangular group*.

Suppose that, for a completely 0-simple semigroup S, a given $\mathcal{H}$-class $H \subseteq S \setminus \{0\}$ is a group, and thus a primitive idempotent $(i, p_{\lambda i}^{-1}, \lambda) \in H$. Since this will be the only idempotent in H, the $\mathcal{H}$-classes excluding $\{0\}$ can be indexed by $I \times \Lambda$, hence $H_{\lambda i}$. Similarly, the $\mathcal{R}$-classes excluding $\{0\}$ can be indexed by I and the $\mathcal{L}$-classes by Λ.

Corollary 5.35 (to Lemma 5.24) Let S be a completely 0-simple semigroup isomorphic to the Rees matrix semigroup $\mathcal{M}^0(G; I, \Lambda; P)$, where G is an infinite group.

(i) If S is left fairly amenable with measure μ then every $\mathcal{R}$-class $R \subseteq S \setminus \{0\}$ has measure given by

$$\mu(R) = \begin{cases} 1/|I| & \text{if } I \text{ is finite} \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If S is right fairly amenable with measure μ then every $\mathcal{L}$-class $L \subseteq S \setminus \{0\}$ has measure

$$\mu(L) = \begin{cases} 1/|\Lambda| & \text{if } \Lambda \text{ is finite} \\ 0 & \text{otherwise.} \end{cases}$$

(iii) Finally, if S is fairly amenable both ways with μ then each $\mathcal{H}$-class $H \subseteq S\setminus\{0\}$ has measure

$$\mu(H) = \begin{cases} 1/|I \times \Lambda| & \text{if both } I, \Lambda \text{ are finite} \\ 0 & \text{otherwise.} \end{cases}$$

□

If an $\mathcal{H}$-class $H_{i\lambda} \subseteq S\setminus\{0\}$ is a group, what group is it? It is isomorphic to G every time. To see this, consider the map $g \mapsto (i, gp_{\lambda i}^{-1}, \lambda)$. It is a bijection as $gp_{\lambda i}^{-1} = hp_{\lambda i}^{-1}$ implies $g = h$, and it is easy to check that the map is a group homomorphism. Therefore we can consider any completely 0-simple semigroup S to be a two-dimensional grid of twisted copies of a group G, with at least enough of the products between $\mathcal{H}$-classes being non-zero for the semigroup to be 0-simple.

To characterise the fairly amenable completely 0-simple semigroups, we need to establish, given a particular element, which sets will that element act injectively on the left of and right of. Since the presence of 0 entries in the sandwich matrix P decreases the sets an element acts injectively on, the completely 0-simple case will be a corollary of the completely simple case.

Lemma 5.36 Let $(i, x, \lambda) \in \mathcal{M}(G; I, \Lambda; P)$. For any map $f : (G \times \Lambda) \to I$, the element (i, x, λ) acts injectively on the left of the set

$$B_{\lambda,f} := \left\{ (f(y, \nu), p_{\lambda,f(y,\nu)}^{-1} y, \nu) : y \in G, \nu \in \Lambda \right\}$$

and this set is maximal. Note that $B_{\lambda,f}$ depends on λ and f, but not i or x, so the subscripts are justified. Hence $\left\{B_{\lambda,f} : f \in I^{G\times\Lambda}\right\}$ is the set of all such maximal sets for a given $\lambda \in \Lambda$. Each $\mathcal{R}$-class $R_i = (i, x, \lambda)B_{\lambda,f}$.

Proof Suppose, for some $y_1, y_2 \in G$ and $\nu_1, \nu_2 \in \Lambda$,

$$(i, x, \lambda)(f(y_1, \nu_1), p_{\lambda,f(y_1,\nu_1)}^{-1} y_1, \nu_1) = (i, x, \lambda)(f(y_2, \nu_2), p_{\lambda,f(y_2,\nu_2)}^{-1} y_2, \nu_2).$$

Then

$$\begin{aligned} (i, xp_{\lambda,f(y_1,\nu_1)} p_{\lambda,f(y_1,\nu_1)}^{-1} y_1, \nu_1) &= (i, xp_{\lambda,f(y_2,\nu_2)} p_{\lambda,f(y_2,\nu_2)}^{-1} y_2, \nu_2) \\ \Rightarrow \quad (i, xy_1, \nu_1) &= (i, xy_2, \nu_2), \end{aligned}$$

thus $y_1 = y_2$ and $\nu_1 = \nu_2$, and so (i, x, λ) acts injectively on the left of $B_{\lambda,f}$.

Now, pick $(j, p_{\lambda j}^{-1}y, \nu) \notin B_{\lambda,f}$. Thus $y \in G$, $\nu \in \Lambda$, $j \in I$, and $f(y, \nu) \neq j$. If there is only one $\mathcal{R}$-class this is impossible, but then $B_{\lambda,f}$ is the whole semigroup and maximality is trivial, so assume there are at least two $\mathcal{R}$-classes. Now,

$$\begin{aligned}(i, x, \lambda)(j, p_{\lambda j}^{-1}y, \nu) &= (i, xp_{\lambda j}p_{\lambda,j}^{-1}y, \nu)\\ &= (i, xy, \nu),\end{aligned}$$

but also

$$\begin{aligned}(i, x, \lambda)(f(y,\nu), p_{\lambda,f(y,\nu)}^{-1}y, \nu) &= (i, xp_{\lambda,f(y,\nu)}p_{\lambda,f(y,\nu)}^{-1}y, \nu)\\ &= (i, xy, \nu),\end{aligned}$$

but since $j \neq f(y, \nu)$, (i, x, λ) does not act injectively on the left of $B_{\lambda,f} \cup \left\{(j, p_{\lambda j}^{-1}y, \nu)\right\}$. Since $(j, p_{\lambda j}^{-1}y, \nu)$ is any arbitrary element not in $B_{\lambda,f}$, $B_{\lambda,f}$ must be maximal. □

Lemma 5.37 The completely simple semigroup $\mathcal{M}(G; I, \Lambda; P)$ is left fairly amenable if I is countably infinite, and right fairly amenable if Λ is countably infinite.

Proof Let $S = \mathcal{M}(G; I, \Lambda; P)$, and suppose I is identified with $\mathbb{N}$ and Λ with a subset of $\mathbb{N}$.

1. Let ξ be any finitely-additive measure on G. Since each $\mathcal{H}$-class $H_{i\lambda}$ of S is an isomorphic copy of G, ξ is easily rewritten as a finitely-additive measure $\xi_{i\lambda}$ on each individual $H_{i\lambda}$ by setting
$$\xi_{i\lambda}\left(\{i\} \times A \times \{\lambda\}\right) := \xi\left(p_{\lambda i}A\right)$$
for every $A \subseteq G$. This may be thought of this as being the same measure ξ, but rotated to line up with $H_{i\lambda}$.
2. Fix free ultrafilters U, V on the sets I, Λ, and then define μ by setting, for all $A \subseteq S$,
$$\mu\left(A\right) := \underset{n\to\infty}{U\text{-}\lim}\,\underset{m\to\infty}{V\text{-}\lim}\frac{1}{nm}\sum_{i=1}^{n}\sum_{\lambda=1}^{m}\xi_{i\lambda}\left(A \cap H_{i\lambda}\right).$$
3. Suppose I is infinite. Then for any set A acted injectively on the left by $(i, x, \lambda) \in S$, $\mu\left(A\right) = 0 = \mu\left((i, x, \lambda)A\right)$, and so μ is left fairly invariant.
   PROOF:
   3.1. Lemma 5.36 provides $\left\{B_{\lambda,f} : f \in I^{G\times\Lambda}\right\}$, the collection of maximal sets acted on the left injectively by (i, x, λ). Every such A is a subset of some $B_{\lambda,f}$, so if $\mu\left(B_{\lambda,f}\right) = 0 = \mu\left((i, x, \lambda)B_{\lambda,f}\right)$ then so to for A.

3.2. For all $i \in I$, $(i, x, \lambda)B_{\lambda,f} = R_i$, an $\mathcal{R}$-class. We have $\mu(R_i) = 0$ since I is infinite, and thus $\mu((i, x, \lambda)B_{\lambda,f}) = 0$ for any $(i, x, \lambda) \in S$ and $B_{\lambda,f}$.

3.3. For each $\nu \in \Lambda$,

$$\begin{aligned}\sum_{j\in I} \xi_{j\nu}(B_{\lambda,f} \cap H_{j\nu}) &= \sum_{j\in I} \xi_{j\nu}\left(\left\{(j, p_{\lambda j}^{-1}y, \nu) : y \in G, j = f(y,\nu)\right\}\right) \\ &= \sum_{j\in I} \xi\left(p_{\nu j}\left\{p_{\lambda j}^{-1}y : y \in G, j = f(y,\nu)\right\}\right) \\ &\leq \xi\left(\bigcup_{j\in I}\{y \in G : j = f(y,\nu)\}\right) \\ &= \xi(G) = 1.\end{aligned}$$

But then $\mu(B_{\lambda,f}) = 0$.

4. The right case holds analogously. □

Corollary 5.38 The completely 0-simple semigroup $\mathcal{M}^0(G; I, \Lambda; P)$ is left fairly amenable if I is infinite, and right fairly amenable if Λ is infinite.

Proof Consider the set $B_{\lambda,f}$ defined in 5.36 and used in Lemma 5.37. It falls short for the completely 0-simple case for two reasons.

(i) If $p_{\lambda j} = 0$ for some $j \in I$, then $(i, x, \lambda)(B_{\lambda,f} \cap R_j) = \{0\}$ so (i, x, λ) fails to act injectively on the left of $B_{\lambda,f} \cap R_j$.

(ii) If there are no such $j \in I$, then $B_{\lambda,f}$ fails to be maximal because $0 \notin (i, x, \lambda)B_{\lambda,f}$.

This can be fixed by defining a substitute set

$$B'_{\lambda,f,z} := \left(B_{\lambda,f} \setminus (i, x, \lambda)^{-1}\{0\}\right) \cup \{z\}$$

where $z \in (i, x, \lambda)^{-1}\{0\}$. Since the left injective action depends on λ but not i, the notation is justified. This new set $B'_{\lambda,f,z}$ can then be substituted for $B_{\lambda,f}$ in the previous proofs. □

Theorem 5.25 and Example 5.27 together demonstrate that if G is amenable then the rectangular group $I \times G \times \Lambda$ is fairly amenable. To extend this result to the $\mathcal{M}(G; I, \Lambda; P)$ case, we can get away with reusing the same finitely-additive measure obtained in Theorem 5.25.

Lemma 5.39 If G is amenable, then $\mathcal{M}(G; I, \Lambda; P)$ is fairly amenable both ways.

Proof By Lemma 5.37, the case where I and Λ are both infinite is already given. Assume, then, that I is finite, so we need to show that the semigroup is left fairly amenable. The right side is shown analogously.

Let μ be the left fairly invariant finitely-additive measure for $I \times G \times \Lambda$, which exists by the virtue of Theorem 5.25. For clarity, since $I \times G \times \Lambda$ and $\mathcal{M}(G; I, \Lambda; P)$ have the same set of elements, let $\circ$ denote the multiplication in the rectangular group $I \times G \times \Lambda$, and $\cdot$ the multiplication in $\mathcal{M}(G; I, \Lambda; P)$, so that μ is left fairly invariant for $\circ$. We wish to demonstrate that it is also fairly invariant for $\cdot$. Note that all of Green's relations are unchanged between the two semigroups, so for example, R_i unambiguously refers to the same $\mathcal{R}$-class.

The element (i, x, λ) acts injectively on the left of $A \subseteq \mathcal{M}(G; I, \Lambda; P)$ if, and only if, for each $j \in I$, $(i, xp_{\lambda j}, \lambda)$ acts injectively on the left of $A \cap R_j \subseteq I \times G \times \Lambda$. To see this, observe that if

$$(i, x, \lambda)(j, y, \nu) = (i, xp_{\lambda j}y, \nu) = (i, xp_{\lambda j}, \lambda) \circ (j, y, \nu)$$

and

$$(i, x, \lambda)(j, z, \xi) = (i, xp_{\lambda j}z, \xi) = (i, xp_{\lambda j}, \lambda) \circ (j, z, \xi)$$

are equal, then $y = z$ and $\nu = \xi$. Suppose (i, x, λ) acts injectively on the left of $A \subseteq \mathcal{M}(G; I, \Lambda; P)$. Then, since I is finite,

$$\begin{aligned}
\mu\left((i, x, \lambda)A\right) &= \mu\left((i, x, \lambda)\bigcup_{j\in I}(A \cap R_j)\right) \\
&= \mu\left(\bigcup_{j\in I}(i, xp_{\lambda j}, \lambda) \circ (A \cap R_j)\right) && \text{by the above} \\
&= \sum_{j\in I} \mu\left((i, xp_{\lambda j}, \lambda) \circ (A \cap R_j)\right) && \text{by finite additivity of } \mu \\
&= \sum_{j\in I} \mu\left(A \cap R_j\right) && \text{by left fair invariance of } \mu \\
&= \mu\left(\bigcup_{j\in I}(A \cap R_j)\right) && \text{by finite additivity of } \mu \\
&= \mu\left(A\right),
\end{aligned}$$

and so μ is left fairly invariant for $\mathcal{M}(G; I, \Lambda; P)$ as it is for $I \times G \times \Lambda$. $\square$

Lemma 5.37 shows that if I and Λ are both infinite, then the semigroup is fairly amenable both ways. Lemma 5.39 above demonstrates that if G is amenable then the semigroup is fairly amenable both ways. The converse of Lemma 5.39 remains to be shown: if the semigroup is fairly amenable both ways and either I or Λ is finite, then G must be amenable.

Lemma 5.40 If the completely simple semigroup $\mathcal{M}(G; I, \Lambda; P)$ is left fairly amenable and I is finite, or it is right fairly amenable and Λ is finite, then G is amenable.

Proof Suppose that $S = \mathcal{M}(G; I, \Lambda; P)$ is left fairly amenable with measure μ, and that I is finite. The right side will follow analogously. By Corollary 5.35, all the $\mathcal{R}$-classes have measure equal to $1/|I| > 0$. Each $\mathcal{R}$-class of S is a subsemigroup of S. Since each $\mathcal{R}$-class has non-zero measure, by Lemma 5.19, it is also fairly amenable both ways. Let μ' be μ scaled up by $|I|$, i.e. $\mu' := |I|\,\mu$, thus for instance $\mu'(R_i) = 1$ for every $i \in I$.

Let σ_i be the relation on R_i given by

$$(i, a, \lambda)\, \sigma_i\, (i, b, \nu) \Leftrightarrow a p_{\lambda i} = b p_{\nu i} \quad \text{for all} \quad (i, a, \lambda), (i, b, \nu) \in R_i.$$

Note that σ_i is a congruence, and that $G \cong R_i/\sigma_i$. Any $(i, x, \lambda) \in R_i$ acts injectively on the left of R_i and hence any subset, but most importantly, if $A \subseteq R_i/\sigma_i$, then

$$\begin{aligned}(i, x, \lambda)\sigma \cdot A &= \{(i, x, \lambda)\sigma \cdot (i, a, \lambda')\sigma : (i, a, \lambda')\sigma \in A\}\\ &= \{(i, x p_{\lambda i} a, \lambda')\sigma : (i, a, \lambda')\sigma \in A\}\\ &= \left\{(i, x, \lambda)(i, a, \lambda') : (i, a, \lambda') \in \bigcup A\right\}\\ &= (i, x, \lambda) \bigcup A,\end{aligned}$$

making σ left convenient as in Definition 5.29. Hence the fair amenability of R_i transfers to $R_i/\sigma \cong G$ by Lemma 5.30. □

Corollary 5.41 The completely simple semigroup $\mathcal{M}(G; I, \Lambda; P)$ is fairly amenable both ways if, and only if, G is amenable or I and Λ are both infinite.

Proof Use Lemmas 5.37, 5.39, and 5.40. □

With Corollary 5.41, fair amenability is characterised for completely simple semigroups. Similar results for completely 0-simple semigroups, having generally smaller maximal fairly invariant sets, such as Corollary 5.38, should follow easily.

Completely regular semigroups further generalise the completely simple groups. A semigroup is completely regular if every $\mathcal{H}$-class is a subgroup. Completely regular semigroups are characterised as being semilattices of completely simple semigroups (Howie, 1995), so the knowledge gleaned above might be easily extended if combined with results about semilattices.

It follows from the results above in this section that one way to construct a semigroup that is left fairly amenable but not right fairly amenable is to start with a non-amenable group, and take the direct product with an infinite left zero semigroup. For demonstrative purposes, such an example follows, corroborating the above results without reference to them.

Example 5.42 Left groups are left simple, right cancellative semigroups that are characterised as being direct products of groups and left zero semigroups. As seen previously, they are also completely simple semigroups with trivial sandwich matrices.

Let Z be the left zero semigroup with elements from $\mathbb{N}$, and let S be the left group $\mathbb{F}_{\{a,b\}} \times Z$. S is left fairly amenable but is not right fairly amenable.

Proof On the left: let ξ be any finitely-additve measure on $\mathbb{F}_{\{a,b\}}$ with $\xi\left(\mathbb{F}_{\{a,b\}}\right) = 1$. ξ is necessarily not invariant. Fix an ultrafilter $\mathcal{U}$ over $\mathbb{N}$ and define the finitely-additive measure μ by setting

$$\mu(A) := \lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^{n} \xi\left(A \cap \left(\mathbb{F}_{\{a,b\}} \times \{k\}\right)\right) \quad \text{for all } A \subseteq S.$$

1. μ exists, is finitely additive, and $\mu(S) = 1$, as usual.
2. μ is left fairly invariant.

   Proof: Suppose $(g, m) \in S$ acts injectively on the left of $A \subseteq S$: since Z is left zero, this implies that $(x, m_1), (x, m_2) \in A \Rightarrow m_1 = m_2$ for all $x \in \mathbb{F}_2$ and $m_1, m_2 \in Z$, so A can be thought of as a map $\mathbb{F}_2 \to Z$. Thus $\mu(A) = 0$. Then,
   $$\begin{aligned}\mu((g, m) \cdot A) &= \lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^{n} \xi\left((g, m)A \cap \left(\mathbb{F}_{\{a,b\}} \times \{k\}\right)\right) \\ &\leq \lim_{\mathcal{U}} \frac{1}{n} \xi\left(\mathbb{F}_{\{a,b\}}\right) \\ &= 0.\end{aligned}$$

On the right: assume S is right fairly invariant with measure ν.

1. A contradiction occurs in a similar manner to the usual proof that $\mathbb{F}_2$ is not amenable.

PROOF: Consider one set of words $F(a) \subset \mathbb{F}_{\{a,b\}}$, which end with the letter a. Then

$$\begin{aligned}(F(a) \times Z) \cdot (a^{-1}, 1) &= (F(a)a^{-1} \times Z)\\ &= S \backslash (F(a^{-1}) \times Z),\end{aligned}$$

and similarly for $F(b)$. Hence

$$\begin{aligned}1 &= \nu(S)\\ &= \nu\left((F(a) \cup F(a^{-1}) \cup F(b) \cup F(b^{-1}) \cup \{1\}) \times Z\right)\\ &\geq \nu(F(a) \times Z) + \nu\left(F(a^{-1}) \times Z\right) + \nu(F(b) \times Z) + \nu\left(F(b^{-1}) \times Z\right)\\ &= \nu\left(F(a)a^{-1} \times Z\right) + \nu\left(F(a^{-1}) \times Z\right) + \nu\left(F(b)b^{-1} \times Z\right) + \nu\left(F(b^{-1}) \times Z\right)\\ &= \nu(S) + \nu(S)\\ &= 2,\end{aligned}$$

contradiction. □

5.2.8 Semigroups with involution

Another result for groups states that every left amenable group is also right amenable, and furthermore, a left invariant measure and right invariant measure can be combined to provide a bi-invariant measure (Wagon, 1993, p148). The techniques of these results do not hold for all semigroups (either classically or fairly). The missing ingredient here is involution.

Lemma 5.43 Let S be a semigroup with involution $*$. If S is left fairly amenable, then it is right fairly amenable (and vice-versa).

Proof We define $A^* := \{a^* : a \in A\}$, and so $(As)^* = s^*A^*$. Suppose that μ on S is left fairly invariant and define ν by setting $\nu(A) = \mu(A^*)$ for all A.

1. ν has total measure 1.

   PROOF:

$$\begin{aligned}\nu(S) &= \nu(S^*)\\ &= \mu(S) = 1.\end{aligned}$$

2. ν is finitely additive.

PROOF: For all disjoint $A, B \subseteq S$,

$$\begin{aligned}\nu(A \cup B) &= \mu((A \cup B)^*) \\ &= \mu(A^* \cup B^*) \\ &= \mu(A^*) + \mu(B^*) \\ &= \nu(A) + \nu(B).\end{aligned}$$

3. ν is right fairly invariant.

PROOF: If s acts injectively on the *right* of A, then for $a, b \in A$,

$$\begin{aligned}s^*a^* = s^*b^* &\Leftrightarrow (as)^* = (bs)^* \\ &\Leftrightarrow as = bs \\ &\Rightarrow a = b \\ &\Leftrightarrow a^* = b^*\end{aligned}$$

and so s^* acts injectively on the *left* of A^*. Then

$$\nu(As) = \mu(s^*A^*) = \mu(A^*) = \nu(A)$$

wherever s acts injectively on the right of A. □

Thus groups, inverse semigroups, semigroups of binary relations, and all other $*$-semigroups join the commutative semigroups as classes of semigroups where each example is either *fairly amenable (both ways)*, or not at all.

5.2.9 Clifford semigroups

In addition to being examples of completely 0-simple semigroups, the 0-groups are examples of *Clifford semigroups*, which are characterised as being strong semilattices of groups (Howie, 1976, p94), and in turn are important examples of inverse semigroups. One wonders, therefore, what we can say about Clifford semigroups in general. Some examples will lead the way.

The following example furnishes us with a fairly amenable Clifford semigroup that is not a 0-group, having a subgroup that fails to be amenable in a non-trivial manner.

Example 5.44 Let S be the union of two groups as follows: set $G \cong \mathbb{F}_2$ (not amenable) and $H \cong \mathbb{F}_1$ (amenable), and let $\phi : G \to H$ be the homomorphism mapping $x \mapsto 1_H$ for all $x \in G$. Define the operation on S as a strong semilattice $Y = (\{1, 0\}, \wedge)$ of the groups G, H, i.e. if one of x or y is in H we map the other via ϕ into H to compute xy. Despite the presence of $\mathbb{F}_2$, S *is* fairly amenable.

Proof Let μ_H witness the amenability of H. Define for S the measure μ given by

$$\mu(A) := \mu_H(\phi(A \cap G) \cup (A \cap H)) \quad \text{for all } A \subseteq S,$$

which is invariant under action of H. Since H is an infinite $\mathcal{H}$-class, $\mu_H(\{1_H\}) = 0$ by Lemma 5.23, and therefore $\mu(G) = 0$. It follows that $\mu(A) = \mu_H(A \cap H)$ for any $A \subseteq S$. If $A \subseteq H$ then $gA = A = Ag$ for all $g \in G$, so μ is trivially invariant under G, and thus μ suffices. □

The following example shows a fairly amenable Clifford semigroup that has no amenable subgroup as part of the semilattice.

Example 5.45 Consider the semilattice on the integers $Y = (\mathbb{Z}, \wedge)$ where $a \wedge b = \min\{a, b\}$ for all $a, b \in Y$, together with a measure μ derived from the Følner sequence given by $F_n = [-n, n] \cap Y$.

Now $\mu(k \wedge Y) = \mu((-\infty, k] \cap Y) = \frac{1}{2}$ for all $k \in Y$, all finite sets have measure 0, and the semilattice is fairly amenable.

Suppose we take S to be a strong semilattice of infinitely many non-amenable groups, as follows:

- Let the semilattice Y be isomorphic to $(\mathbb{Z}, \wedge)$, as previously.
- Then, for each $k \in \mathbb{Z}$, let G_k be a non-amenable group.
- Finally, for each $k \in \mathbb{Z}$ let ν_k be any finitely-additive measure on G_k with $\nu_k(G_k) = 1$ (which is necessarily not invariant).

We can extend the μ given on Y to a fairly-invariant μ_S on S by setting, for a fixed free ultrafilter $\mathcal{U}$ over $\mathbb{N}$,

$$\mu_S(A) = \lim_{\mathcal{U}} \frac{1}{2n+1} \sum_{k=-n}^{n} \nu_k(G_k \cap A).$$

While every G_k is not amenable, μ_S witnesses the fair amenability of S. □

Corollary 5.46 If the Clifford semigroup S is a strong *finite* semilattice Y of groups and S is fairly amenable, at least one of the groups is amenable.

Proof Suppose all the groups in $\{G_y : y \in Y\}$ are non-amenable, and the finitely-additive measure μ witnesses the fair amenability of S.

1. $\mu(G_y) = 0$ for all $y \in Y$.
   PROOF: Use Lemma 5.19.
2. $1 = \mu(S) = 0$, contradiction.
   PROOF: $S = \bigcup_{y \in Y} G_y$, which is a disjoint union, and then as there are only finitely many groups in the semilattice, $\mu(S) = 0$. □

5.2.10 Directed unions of semigroups

One final theorem on groups that translates well to fairly amenable semigroups is that a directed union of amenable groups is also amenable.

Theorem 5.47 If S is the directed union of left [right] fairly amenable semigroups, then S is left [right] fairly amenable.

Proof This proof uses essentially the same topological argument as in Wagon (1993, p150). Let $\{S_i : i \in I\}$ be the directed system of left fairly amenable semigroups whose union is S: i.e. for each $a, b \in I$ there exists a $c \in I$ such that S_a and S_b are sub-semigroups of S_c, and, $S = \bigcup_{i \in I} S_i$. For each $i \in I$:

- let μ_i be the left fairly invariant finitely-additive measure corresponding to S_i, and
- let M_i be the set of finitely-additive measures $m : \mathcal{P}(S) \to [0, 1]$ such that $m(S) = 1$ and whenever $s \in S_i$ acts injectively on $A \subseteq S$, $m(sA) = m(A)$.

1. M_i is non-empty for all $i \in I$.
   PROOF: Define $m_i(A) := \mu_i(A \cap S_i)$ for all $A \subseteq S$. Clearly $m_i \in M_i$.
2. Each M_i is a closed subset of $[0, 1]^{\mathcal{P}(S)}$.
   PROOF: Suppose $f \notin M_i$. Then f fails to be finitely additive, fails to be left fairly invariant for some $s \in S_i$, or $f(S) \neq 1$. It is possible to take f and vary the amount ϵ by which any of the three conditions is violated to obtain some nearby f_ϵ, by adding to or subtracting from the value f takes at one or more points in $\mathcal{P}(S)$. For example, if $f(S) = k \neq 1$, then we can find a similar function f_ϵ given by $f_\epsilon(S) := k + \epsilon \neq 1$, $f_\epsilon(A) := f(A)$ for $A \neq S$. Thus each $f \notin M_i$ is contained in an open neighborhood (however small) consisting of points that fail to have one of the three conditions. The union of all these open neighborhoods is open, and the complement is M_i, which is therefore closed. This argument is essentially the same as Wagon (1993, p126).

3. The collection $\{M_i : i \in I\}$ has the finite intersection property.
   Proof: If $S_a, S_b \subseteq S_c$ then $M_a \cap M_b \supseteq M_c$, since each member must be left fairly invariant for increasingly many elements.
4. There exists some $\mu \in \bigcap_{i \in I} M_i$ which is the required left fairly-invariant measure.
   Proof: From Tychonoff's Theorem, the space $[0, 1]^{\mathcal{P}(S)}$ is compact. Compactness is equivalent to the property that any collection of closed subsets with the finite intersection property is nonempty, and $\{M_i : i \in I\}$ is an example of such a collection.

The right case is handled analogously. □

Just as with Abelian groups, every Abelian semigroup is the direct union of its finitely-generated Abelian subsemigroups, and therefore we almost have enough to show all Abelian semigroups are fairly amenable.

5.3 Further examples

Proposition 5.48 Any finitely-generated free Abelian semigroup, such as $(\mathbb{N}, +)$, is fairly amenable.

Proof The free Abelian semigroup on k generators is isomorphic to $(\mathbb{N} \cup \{0\})^k$ minus the origin, and again every action is injective. The Følner sequence given by $F_n = \{(a_1, a_2, \ldots, a_k) : a_1, a_2, \ldots a_k < n\}$ suffices. □

Proposition 5.49 The semigroup of natural numbers with *multiplication*, $(\mathbb{N}, \cdot)$ is also a cancellative Abelian semigroup. However, unlike the previous example, it is infinitely generated—by the primes. Fortunately, it is also fairly amenable.

Proof As usual a totally invariant finitely-additive measure is required. There exists a Følner sequence $\{F_n\}_{n \in \mathbb{N}}$ where F_n consists of the products of powers of the first n primes, and each power lies in $[0, n]$, i.e.

$$F_n := \left\{ p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n} : 0 \leq i_j \leq n, j = 1, \ldots, n \right\},$$

as required. Bergelson (2005) demonstrated a family of Følner sequences of this kind. □

Example 5.50 The free semigroup on two generators $FS_2 = \{a, b\}^+$, and the free monoid on two generators, are neither left nor right fairly amenable.

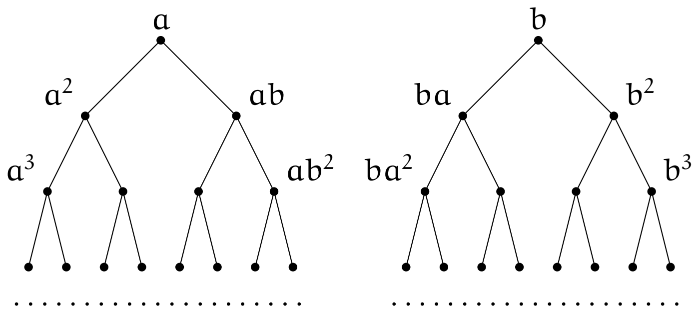

Figure 5.3: The right Cayley graph for the free semigroup on two generators $\{a, b\}^+$.

Proof Suppose $S = \{a, b\}^+$ is left fairly amenable and μ is the required measure. Note that a and b both act injectively on S and so we require $\mu(aS) = \mu(S) = \mu(bS)$. But since $S = \{a, b\} \cup aS \cup bS$,

$$1 = \mu(S) = \mu(\{a, b\}) + \mu(aS) + \mu(bS) = \mu(\{a, b\}) + 1 + 1 \geq 2,$$

contradiction. By a similar argument, FS_2 is not right fairly amenable. (Alternatively, endow the semigroup with an involution $*$ where $a^* := b$ and vice-versa, and apply Lemma 5.43.) $\square$

Remark 5.51 Note that the previous argument can be adapted to any finite number of generators $n \geq 2$. Note also that FS_2^0 (the free semigroup on two generators with a zero adjoined) is now not fairly amenable either, in contrast to the classical case.

5.3.1 Graph inverse semigroups

Example 5.52 The bicyclic monoid B is fairly amenable.

Proof Recall that $B = \text{mon}\langle p, q | pq = 1\rangle = \{q^m p^n : m, n \in \mathbb{N} \cup \{0\}\}$.

Consider the sequence given by $\square_n = \left\{q^j p^k : j, k \leq n\right\}$ for all $n \in \mathbb{N}$. It will suffice to show this sequence is Følner for any element on the left.

The element q acts injectively on the left of all B, so $|q\square_n| = |\square_n|$ and $|q\square_n \triangle \square_n| = 2n$. p on the other hand does not act injectively on the left of $\square_n$, in which case $|p\square_n| \leq |\square_n|$. Since the minimal non-injective sets for each left multiplication by p are $\left\{p^k, qp^{k+1}\right\}$ for each k, we can see exactly that $|p\square_n| = (n-1)n + 1$, and

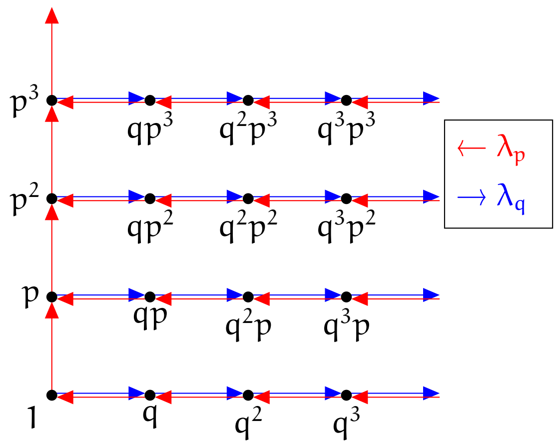

Figure 5.4: Part of the left Cayley graph of the bicyclic monoid B.

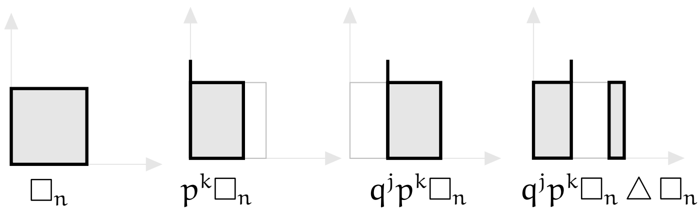

Figure 5.5: Deriving $\left|q^jp^k\square_n \triangle \square_n\right|$ in the bicyclic monoid.

$|p\square_n \triangle \square_n| = n+1$. For any arbitrary $x = q^jp^k$, then,

$$|x\square_n \triangle \square_n| = k + n(2j-k) \quad \text{for all } n > j$$

(depicted in Figure 5.5) which is linear in n, and therefore the Følner sequence $\{\square_n\}_{n\in\mathbb{N}}$ suffices.

Since B is inverse, by Lemma 5.43 B is fairly amenable on both sides. $\square$

Example 5.53 The polycyclic monoid on two generators, P_2, is not fairly amenable. As described by Milan (2008), P_2 and every other graph inverse semigroup has the weak containment property, so it follows that fair amenability is not equivalent to weak containment.

Proof Recall that

$$P_2 = \mathrm{mon}^0 \left\langle p, q, p^{-1}, q^{-1} \middle| pp^{-1} = 1 = qq^{-1}, pq^{-1} = 0 = qp^{-1} \right\rangle,$$

and so every element other than 0 or 1 can be written canonically in the form $x^{-1}y$, where x, y are possibly empty strings over the alphabet $\{p, q\}$ (Lawson, 2004). It follows that at least the free monoids $\left\{p^{-1}, q^{-1}\right\}^*$ and $\{p, q\}^*$ are embedded within P_2.

Perhaps we could try to show that fair amenability is impossible for P_2 by showing that one of these copies is the result of taking the quotient of P_2 by a left convenient congruence to obtain $\{p, q\}^*$, but this is not as clear as the following.

1. Assume P_2 is left fairly amenable with measure μ, and for each $x \in P_2$ let $H_x \subseteq P_2$ consist of elements with their canonical form starting with the string x. P_2 can be decomposed like so:
$$P_2 = H_{p^{-1}} \uplus H_{q^{-1}} \uplus H_p \uplus H_q \uplus \{0, 1\}.$$
2. Consider the injective left actions $\lambda_{p^{-1}}, \lambda_{q^{-1}}$
$$p^{-1}P_2 = H_{p^{-1}} \uplus \{0\}, \quad q^{-1}P_2 = H_{q^{-1}} \uplus \{0\}.$$
3. Apply μ to see that it is not left fairly invariant.

PROOF:
$$\begin{aligned}
1 &= \mu(P_2) \\
&= \mu\left(H_{p^{-1}} \uplus H_{q^{-1}} \uplus H_p \uplus H_q \uplus \{0, 1\}\right) \quad \because \text{step 1} \\
&= \mu\left(H_{p^{-1}}\right) + \mu\left(H_{q^{-1}}\right) + \mu(H_p) + \mu(H_q) + \mu(\{0, 1\}) \\
&= \mu\left(H_{p^{-1}}\right) + \mu\left(H_{q^{-1}}\right) + \mu(H_p) + \mu(H_q) \quad \because \text{Lemma 5.14} \\
&= \mu\left(p^{-1}P_2\right) + \mu\left(q^{-1}P_2\right) + \mu(H_p) + \mu(H_q) \quad \because \text{step 2} \\
&= 1 + 1 + \mu(H_p) + \mu(H_q) \quad \because \text{fair invariance} \\
&\geq 2,
\end{aligned}$$
contradiction.

As with the bicyclic monoid, P_2 is also inverse, so by Lemma 5.43 it is not right fairly amenable either. $\square$

Remark 5.54 As with FS_2 and greater, the previous argument can be adapted to any finite number of generators $n \geq 2$. P_2 is also an example of an inverse semigroup that is not fairly amenable, but is classically amenable because the maximal group homomorphic image (the trivial group) is amenable.

Polycyclic monoids were shown by Lawson (1998, pp289-292) to have the following properties.

(i) P_n is *combinatorial*, i.e. the relation $\mathcal{H}$ is the equality relation,

(ii) P_n is E^*-unitary, and 0-bisimple.

(iii) For $n \geq 2$, P_n is also congruence free.

(iv) P_n is embedded in P_2 for every $n \geq 2$, including $n = \infty$.

Both the bicyclic and polycyclic monoids above are generalised by the *graph inverse semigroups*, of which a very rapid definition is given below. It is not necessary here to refresh ourselves on the complete history of graph inverse semigroups. Some exposition was given by Jones and Lawson (2011). However, there are interesting parallels between graph inverse semigroups and Cuntz-Krieger C*-algebras.

Definition 5.55 Let G be a directed graph, with multiple edges between pairs of vertices allowed, and let G^* be the free category over G—that is, G^* is the set of all paths in G together with concatenation where sensible in G. The *graph inverse semigroup* P_G is the semigroup with zero with elements $(G^* \times G^*) \cup \{0\}$ and operation given by setting for all $x, y, u, v \in G^*$,

$$(x, y)(u, v) := \begin{cases} (xz, v) & \text{if } \exists z \in G^* \text{ such that } u = yz \\ (x, zv) & \text{if } \exists z \in G^* \text{ such that } y = uz \\ 0 & \text{otherwise.} \end{cases}$$

If $(x, y)(u, v) \neq 0$ then we say the two elements are *prefix compatible*, i.e. the required z above. As a notation initially, we write $(x, y) \in P_G$ as xy^{-1}. Note that if $u = yz$ then in this notation, $xy^{-1} \cdot uv^{-1} = xy^{-1}yzv^{-1} = xzv^{-1}$, so effectively $y^{-1}y = 1$ is a rule in the presentation.

The 1-rose generates as its free category the free monoid on one generator, and the n-rose generates the free monoid on n generators in the same way. The bicycylic monoid B is the graph inverse semigroup P_G where G is a 1-rose, with the zero removed. The polycyclic monoid P_n is the graph inverse semigroup over an n-rose. The free monoid infrastructure of P_2 is what led to Example 5.53.

Remark 5.56 Graph inverse semigroups P_G have the following properties.

(i) P_G is finite if, and only if, G is a finite tree.

(ii) P_G is combinatorial.

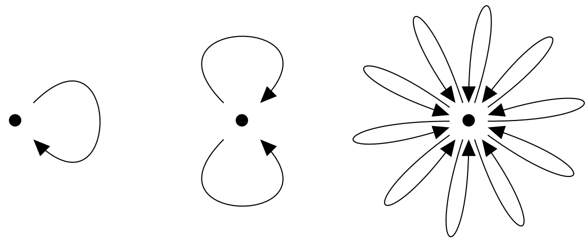

Figure 5.6: From the left, the 1-rose, 2-rose, and 10-rose. The free categories over these are simply the free monoids on 1 generator, 2 generators, and 10 generators, respectively.

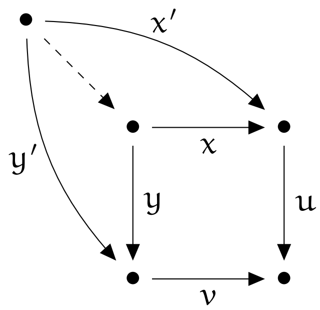

Figure 5.7: The arrows x, y, u, and v form a commutative square. If for any other pair of arrows x' and y' with common domain form a commutative square with u and v there exists a unique arrow from $\mathbf{d}(x') \to \mathbf{d}(x)$ (dashed), then x and y are a *pullback* of u and v.

(iii) $E(P_G)$ consists only of elements of the form xx^{-1}.

(iv) The natural partial order on P_G is characterised by $xy^{-1} \leq uv^{-1} \Leftrightarrow x = up$ and $y = vp$ for some p.

(v) P_G is 0-bisimple only if it is a polycyclic monoid.

Graph inverse semigroups are themselves a special subclass of the inverse semigroup built on left cancellative Leech categories, as characterised by Jones and Lawson (2011). A brief account of the definitions is given below.

A *Leech category* C is a category in which each pair of arrows with common range, that can be completed to form a commutative square, has a pullback. For a brief explanation of these terms, see Figure 5.7. Free categories generated by graphs are

$$\begin{tikzcd}
\bullet \arrow[r, dashed, "x"] \arrow[d, dashed, "y"'] & \bullet \arrow[r, "a"] \arrow[d, "b"] & \bullet \\
\bullet \arrow[r, "c"'] \arrow[d, "d"'] & \bullet & \\
\bullet & &
\end{tikzcd}$$

Figure 5.8: The product of two elements in $(C \times C)/\sim$, $[a, b]$ and $[c, d]$, is defined to be $[ax, dy]$ when there exists a pullback x, y of b, c, and 0 if there is no such pullback.

examples of left cancellative Leech categories.

Given a left cancellative Leech category C, how does one construct an inverse semigroup $S(C)$ generalising the graph inverse semigroups? Let $\sim$ be the relation on $C \times C$ given by

$$(a, b) \sim (a', b') \Leftrightarrow \exists u \in C \text{ such that } (a, b) = (a', b')u.$$

Then $S(C) = ((C \times C)/\sim) \cup \{0\}$ with the multiplication of $\sim$-classes $[a, b]$ and $[c, d]$ described in Figure 5.8. Here, the pullback arrows, which always exist wherever commutative squares exist since C is a Leech category, take the place of prefix-compatibility.

Question 5.57 For graph inverse semigroups, or even $S(C)$-style inverse semigroups, can the boundary between the left fairly amenable and non-left fairly amenable be refined, possibly in terms of the graph G in P_G, or the left cancellative Leech category C in $S(C)$?

5.3.2 The Baer-Levi semigroup

Definition 5.58 The *Baer-Levi semigroup* $BL(p, q)$ is defined as being the set of injective maps f, on some infinite set X with $|X| = p$, where f has the property that $|X \backslash f(X)|$ is the fixed infinite cardinal $q \leq |X|$, together with the composition operation (Clifford and Preston, 1967).

Conventionally, products in composition semigroups of maps such as the Baer-Levi semigroups are written in "algebraist" order—the composition of f and g is written fg. With that ordering, Baer-Levi semigroups are right cancellative and right simple. However, to remain consistent with the treatment of other functions in this thesis, I shall deviate from this convention and denote the product with $\circ$, making the Baer-Levi semigroups *left* cancellative and *left* simple.

Example 5.59 For a Baer-Levi semigroup $BL(p, q)$,

(i) $BL(p, q)$ is not left fairly amenable if $p = q$, and

(ii) $BL(p, q)$ is not right fairly amenable for all p, q.

Proof For succinctness let S be shorthand for $BL(p, q)$. On the left:

1. Let $a, b \in S$ be such that the right ideals $a \circ S$ and $b \circ S$ are disjoint. (There are two disjoint right ideals if, and only if, $p = q$.) For example, if S is the Baer-Levi semigroup on $\mathbb{N}$, we may pick $a : n \mapsto 2n$ and $b : n \mapsto 2n + 1$. Let $R = S \backslash ((a \circ S) \cup (b \circ S))$.
2. Since S is left cancellative, every left action is injective.
3. Assume S is left fairly amenable with measure μ, and derive a contradiction.

   Proof:

$$\begin{aligned} 1 &= \mu(S) \\ &= \mu((a \circ S) \,\dot\cup\, (b \circ S) \,\dot\cup\, R) && \text{by defintion} \\ &= \mu(a \circ S) + \mu(b \circ S) + \mu(R) \\ &= \mu(S) + \mu(S) + \mu(R) && \because \text{ left fairly invariant} \\ &\geq 2, \end{aligned}$$

   a clear contradiction.

On the right:

1. For each $s \in S$ let the equivalence relation θ_s be given by $a\, \theta_s\, b \Leftrightarrow a \circ s = b \circ s$ for all $a, b \in S$. Since S consists of maps on the set X, θ_s depends only on $s(X)$, so $a\, \theta_s\, b \Leftrightarrow a|_{s(X)} = b|_{s(X)}$.

PROOF: For any $a, b, s \in S$,

$$\begin{aligned} a\,\theta_s\,b &\Leftrightarrow a \circ s = b \circ s \\ &\Leftrightarrow a\,(s\,(x)) = b\,(s\,(x)) \text{ for all } x \in X \\ &\Leftrightarrow a\,(y) = b\,(y) \text{ for all } y \in s\,(X) \\ &\Leftrightarrow a|_{s(X)} = b|_{s(X)}. \end{aligned}$$

2. For every $s \in S$, every θ_s-equivalence class is nonempty and infinite.

   PROOF: By definition $|X \backslash s\,(X)|$ is some infinite cardinal, therefore a Baer-Levi semigroup on $X \backslash s\,(X)$ can be used to generate elements of each θ_s-class.

3. For each $s \in S$ define two disjoint sets M_1, M_2 by choosing two distinct elements from each θ_s-class. $S \circ s = M_1 \circ s = M_2 \circ s$ and while the action $S \circ s$ is not injective, the actions on M_1 and M_2 are injective.

   PROOF: By definition, θ_s partitions S into sets that map to the same element under the right action of s, so $S \circ s = M_1 \circ s = M_2 \circ s$. For any $a, b \in M_1$, $a \circ s = b \circ s \Rightarrow a\,\theta_s\,b \Rightarrow a = b$, similarly for M_2.

4. Assume that S is right fairly amenable with measure ν. This results in a contradiction.

   PROOF: Let $R = S \backslash (M_1 \cup M_2)$, then

$$\begin{aligned} 1 &= \nu\,(S) \\ &= \nu\,(M_1 \uplus M_2 \uplus R) \quad \because \text{definition} \\ &= \nu\,(M_1) + \nu\,(M_2) + \nu\,(R) \\ &= \nu\,(M_1 \circ s) + \nu\,(M_2 \circ s) + \nu\,(R) \quad \because \text{right fairly invariant} \\ &= \nu\,(S \circ s) + \nu\,(S \circ s) + \nu\,(R) \quad \because \text{step 3} \\ &= \nu\,(S) + \nu\,(S) + \nu\,(R) \quad \because S \text{ is left simple} \\ &= 1 + 1 + \nu\,(R) \\ &\geq 2, \end{aligned}$$

   a clear contradiction. □

5.3.3 Free inverse semigroups

Example 5.60 The free inverse semigroup on one generator FIS_1 is fairly amenable both ways.

Proof From Munn's Theorem on the structure of free inverse semigroups (Lawson, 1998), elements of FIS_1 can be thought of as triples of integers

$$FIS_1 \cong \left\{(p,q,r) \in \mathbb{Z}^3 : p \geq 0, p+q \geq 0, q+r \geq 0, r \geq 0, p+q+r \geq 0\right\}$$

with the product defined by

$$(p,q,r)(p',q',r') := (\max\{p, p'-q\}, q+q', \max\{r', r-q'\})$$

for all $(p,q,r),(p',q',r') \in FIS_1$ (Lawson, 1998, p193). Consider the increasing sequence given by

$$F_n = \{(x,y,z) \in FIS_1 : x,y,z \leq n\}.$$

1. The sequence $\{|F_n|\}_{n\in\mathbb{N}}$ is the sequence of "house numbers"—imagine a filled cube "house" of height n, with a filled rectangular pyramid "roof" on top (for further insights into this fascinating integer sequence, see the On-line Encyclopedia of Integer Sequences sequence A051662)—a sequence which is given by
$$|F_n| = (n+1)^3 + \frac{1}{6}n(n+1)(2n+1) \quad \text{for all } n,$$
and thus $(n \mapsto |F_n|) \in O(n^3)$.
2. Let $(p,q,r) \in FIS_1$. By definition,
$$(p,q,r)F_n = \{(\max\{p, x-q\}, q+y, \max\{z, r-y\}) : (x,y,z) \in F_n\}.$$
3. For large n,
$$|(p,q,r)F_n| \approx |\{(x-q, q+y, z) : (x,y,z) \in F_n\}|$$
i.e. the left action of (p,q,r) on F_n is an almost-translation in $\mathbb{Z}^3$, and in particular
$$\begin{aligned}|F_n \triangle (p,q,r)F_n| &\approx |F_n \triangle \{(x-q, q+y, z) : (x,y,z) \in F_n\}| \\ &\approx 2qn^2.\end{aligned}$$
Thus $(n \mapsto |F_n \triangle (p,q,r)F_n|) \in O(n^2)$, and therefore the sequence $\{F_n\}_{n\in\mathbb{N}}$ is Følner. The right case holds similarly. □

5.3.4 Summary

Some of the examples and results from the previous sections are summarised in Table 5.1. The variety of interesting examples demonstrate that the "fair" modification of invariant finitely-additive measures interacts well with the structure of semigroups. Some important results from group amenability theory are preserved, and examples of fairly amenable semigroups, especially with zeroes, are more gratifying.

Kind of semigroup	Classically amenable	Fairly amenable
Finite	$\Leftrightarrow$ Unique min. ideals	Yes (5.12)
With zero	Yes	Sometimes (5.20)
Monogenic	Yes	Yes (5.48)
Free ($\geq$ 2 gen.)	No	No (5.50)
Abelian	Yes	?
Clifford	Sometimes	Sometimes (5.44)
Left/right zero sgp	Sided	Yes (5.26)
Left/right group	?	Sometimes (Sided; 5.42)
Completely 0-simple	Yes ($\because$ zero)	Sometimes (§5.2.7)
Baer-Levi	?	No (5.59)
Inverse S	$\Leftrightarrow$ $G(S)$ is	?
Bicyclic	Yes	Yes (5.52)
Polycyclic	Yes ($\because$ zero)	No (5.53)
Graph inverse	Almost always ($\because$ zero)	?
Free monogenic inverse	Yes	Yes (5.60)

Table 5.1: Amenability versus fair amenability on different semigroups.

The given examples of non-fairly amenable semigroups have a certain self-similarity which might be used to create Banach-Tarski-style paradoxes.

5.4 Motivations for fair amenability

A particularly nice justification for fair amenability will be shown in the next chapter. Some intuitive justifications for considering fair amenability are given now.

In groups, Definition 2.1 captures philosophically a generalisation of the ratio in finite groups G given by

$$\mu(A) = \frac{|A|}{|G|},$$

or, put another way, $\mu(A)$ is "the fraction of space A occupies within G." Expecting μ to be invariant under a group action is then a generalisation of the idea that "size" of a set remains unchanged under a group action, which in the finite case is bijective—the cardinality of a set is invariant, so too should its "size" be. This principle is implemented sufficiently well for all amenable groups using a limit of finite densities (see Lemma 2.15).

In semigroups, a (self-)action is potentially *any arbitrary transformation*. In particular, multiple elements become "collapsed" together, so the cardinality of a finite set is usually reduced under the action: Consider the actions of an element s on any set A, in any semigroup S. The action of s on A is a surjection $A \to sA$, so

$$|sA| \leq |A| \quad \text{and} \quad |As| \leq |A|.$$

This is the motivation for sub-invariance. If s acts injectively on A—and since it is already surjective, bijectively—then

$$|sA| = |A| \quad [|As| = |A|].$$

This is the motivation for fairly invariant measures.

In finite groups S, $|sA| = |A|$ $[|As| = |A|]$ if, and only if, s acts injectively on the left [right] of A.

Suppose that $A = B \uplus C$, in which case $|A| = |B| + |C|$. If s acts injectively on B and C but not A, then $sB \cap sC \neq \emptyset$, and $|sA| = |sB| + |sC| - |sB \cap sC|$.

Note that if s acts injectively on A, then it acts injectively on every subset of A, but if s acts injectively on A and B individually it does not necessarily act injectively on $A \cup B$. There is, for any set A that s acts on injectively, some maximal set(s) containing A that s acts on injectively.

* * *

Another interpretation is that μ corresponds to a probability distribution for a stochastic process on S considered as a state space, and λ the conditional probability. Consider $\mu(A) = P(x \in A)$ where x is randomly chosen from S (with uniform distribution). Clearly μ is a finitely-additive probability measure. $y \in sA$ if and only if $\exists x : y = sx$ and $x \in A$, so we get

$$P(y \in sA) = P(\exists x : y = sx \& x \in A) = P(x \in A) \cdot P(y = sx | x \in A)$$

or in terms of μ, $\mu(sA) = \mu(A) \cdot \lambda(s, A)$, where λ is interpreted as the conditional probability, thus obtaining $\lambda(s, A) = 1$ whenever s acts injectively on A.

5.4.1 Measure ratios

So without the interpretation as a probability, what is λ from the previous section? van Douwen (1992) studied "nice" measures on $\mathbb{N}$ in the sense that "if Y is obtained from X by a process which makes sets r times as small, from an intuitive point of view, then $\mu(Y) = r^{-1}\mu(X)$ for all X, Y", so, λ is similar to r^{-1} from that perspective. However, the two differ. r^{-1} is a value introduced for the purposes of calculating $\mu(sA)$ given $\mu(A)$, whereas, $\lambda(s, A)$ is better thought of as a quantity calculated after obtaining both $\mu(A)$ and $\mu(sA)$, which are .

Definition 5.61 Let S be a semigroup and μ a finitely-additive finite measure (not necessarily invariant in any way). Define, for l_μ and r_μ both $S \times \mathcal{P}(S) \to \mathbb{R}$,

$$l_\mu(s, A) = \frac{\mu(sA)}{\mu(A)}, \quad r_\mu(s, A) = \frac{\mu(As)}{\mu(A)}$$

and setting $l_\mu(s, A) = 1$ [$r_\mu(s, A) = 1$] when $\mu(A) = 0$.

These quantities are quite obviously related to measures, and for cases where the measure is invariant, these ratios take the value 1.

Lemma 5.62 If S is left [right] fairly amenable with μ, then $l_\mu(s, A) \in [0, 1]$ [$r_\mu(s, A) \in [0, 1]$] for any $s \in S, A \subseteq S$, furthermore, $l_\mu(s, A) = 1$ [$r_\mu(s, A) = 1$] whenever s acts injectively on A.

Proof The $\mu(A) = 0$ case is trivial, so assume $\mu(A) > 0$. Then $\mu(sA) \leq \mu(A)$ so $\mu(sA)/\mu(A) \leq 1$, and also $\mu(sA) \geq 0$ so $\mu(sA)/\mu(A) \geq 0$. If s acts injectively on A then $\mu(sA) = \mu(A)$ and therefore $\mu(sA)/\mu(A) = 1$. Similarly on the right. $\square$

In particular, if S is a monoid then $l_\mu(1, A) = 1 = r_\mu(1, A)$ for all finitely-additive measures μ and sets $A \subseteq S$.

Corollary 5.63 Let S be a left [right] fairly amenable semigroup with zero. 0 acts injectively only on singleton sets, so for any non-empty subset A,

$$l_\mu(0, A) = \frac{1}{|A|}[= r_\mu(0, A)]$$

(0 if A is infinite, 1 if A is a singleton).

Lemma 5.64 Let S be a left [right] fairly amenable semigroup and $A \subseteq S$ such that $A\mu > 0$. Associativity implies $(ab)A = a(bA)$ and $A(ab) = (Aa)b$ for any $a, b \in S, A \subseteq S$, so

$$l_\mu(ab, A) = l_\mu(a, bA) \cdot l_\mu(b, A) \quad [r_\mu(ab, A) = r_\mu(b, Aa) \cdot r_\mu(a, A)]$$

and for any finite product,

$$l_\mu(s_1 s_2 \cdots s_n, A) = \prod_{i=1}^{n} l_\mu(s_i, (s_{i+1} s_{i+2} \cdots s_n) A).$$

Some other easy results relate to idempotents and units.

Corollary 5.65 Let S be a left [right] fairly amenable semigroup and $e \in S$ be an idempotent. $eS = e^2 S$ $[Se = Se^2]$, so $l_\mu(e, A) = 1 [= l_\mu(e, A)]$ for any $A \subseteq eS$ $[Se]$ with $\mu(A) > 0$.

Corollary 5.66 Let u be a unit in a left [right] fairly amenable monoid S, and $A \subseteq S$. By Lemma 5.64,

$$\begin{aligned} 1 &= l_\mu(1, A) \\ &= l_\mu\left(u^{-1}u, A\right) \\ &= l_\mu\left(u^{-1}, uA\right) \cdot l_\mu(u, A) \end{aligned}$$

and given that l_μ and r_μ are restricted to values in $[0, 1]$, we have $l_\mu(u, A) = 1 [= r_\mu(u, A)]$ for any unit u. □

Chapter 6

Fairly Invariant Means for Semigroups

In this chapter we extend fairly-invariant measures to the context of means. This is not particularly challenging.

Let S be a semigroup, $s \in S$, and $A \subseteq S$. The ordinary left action of s on χ_A satisfies $s \cdot \chi_A = \chi_{s^{-1}A}$, since $(s \cdot f)(t) = f(st)$ for all $t \in S$, and by definition, $st \in A \Leftrightarrow t \in s^{-1}A$. Hence, a semigroup S is classically left amenable if, and only if, there is a finitely-additive probability measure μ satisfying $\mu(A) = \mu\left(s^{-1}A\right)$ for all $s \in S$ and $A \subseteq S$ (Paterson, 1988, Exercise 0.32).

Though the dual left and right actions are always well-defined for $\ell^\infty(S)$ and $xf \in \ell^\infty(S)$ for any $x \in S$ and $f \in \ell^\infty(S)$, they have, in the case of a non-injective element, the effect of "flattening" sections or all of the function being acted upon, for instance, see Figure 6.1. In particular, the dual left action is not closed in $\ell^1(S)$. For these reasons it is worth exploring alternative actions, and to find the appropriate analogue of fair invariance such exploration is necessary.

We need not look far. Recall that the left fair amenability condition introduced in the previous chapter is couched as the existence of a finitely-additive probability measure μ satisfying $\mu(A) = \mu(sA)$ wherever the left-action is injective. Therefore, when reduced to means of indicator functions, the left action and mean together must satisfy, under some set of conditions, $m(\chi_A) = m(s \cdot \chi_A) = m(\chi_{sA})$. Viewed through the lens of classical amenability, it seems that the action $\cdot$ should satisfy $s \cdot \chi_A = \chi_{sA}$ for every set A we want to measure, but as we shall see, this need not be the case. Convolution can be used.

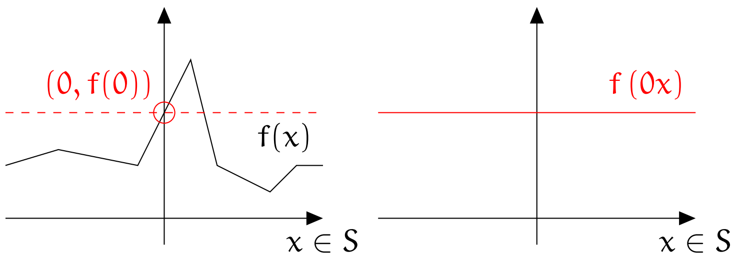

Figure 6.1: The result of the dual left action of 0 on some $f \in \ell^\infty(S)$.

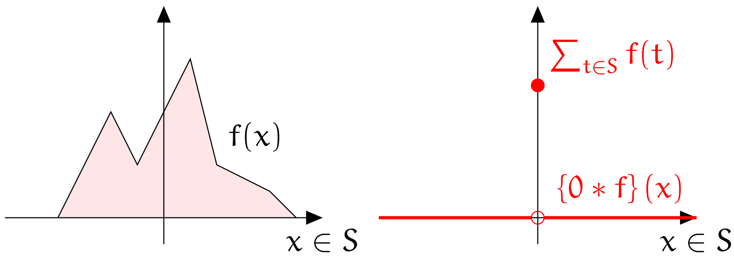

Figure 6.2: The result of the left $*$-action of 0 on some $f \in \ell^1(S)$.

For real- or complex-valued functions $f : S \mapsto \mathbb{K}$ let the support of f be denoted $\mathrm{supp}(f)$, i.e.

$$\mathrm{supp}(f) := \{x \in S : f(x) \neq 0\}.$$

When two functions f and g have disjoint support (i.e. $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$), we will simply say f and g are *disjoint*.

6.1 The convolution partial action

Recall that convolution of two functions $f, g \in \ell^1(S)$, denoted $f * g$, is defined by setting

$$\{f * g\}(x) := \sum_{st=x} f(s)\, g(t) \quad \text{for all } x \in S.$$

This extends to a left convolution "action" of $s \in S$ on $f \in \ell^\infty(S)$, denoted $s * f$, which may be defined by setting

$$\{s * f\}(x) := \sum_{st=x} f(t) \quad \text{for all } x \in S.$$

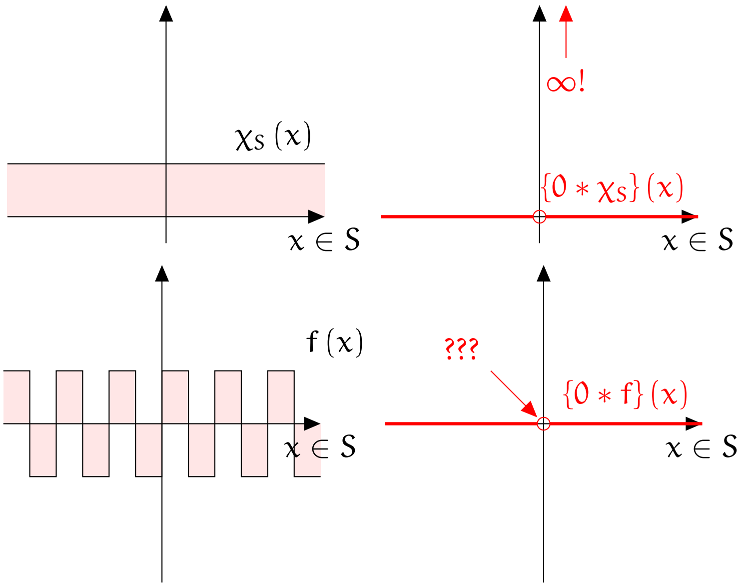

Figure 6.3: Some example cases where the convolution partial action of 0 is not well-defined on $\ell^\infty(S)$.

Alternatively,

$$\{s * f\}(x) = \sum_{t \in s^{-1}x} f(t) \quad \text{for all } x \in S.$$

For each $s \in S$, let the equivalence relation θ_s on S be given by setting $x \,\theta_s\, y$ if and only if $sx = sy$, for all $x, y \in S$. Note that each $s^{-1}x$ is precisely a θ_s-equivalence class.

Unsurprisingly, $*$ often fails to be an operation that is closed in $\ell^\infty(S)$, or even well-defined. In contrast to the dual action which "flattens" along sections of the domain (see Figure 6.1), the convolution "action" has the appearance of "bunching up" the values along the domain (Figure 6.2). For an extreme example, suppose S is an infinite semigroup with zero. Then

$$0 * \chi_S = \sum_{t \in S} \chi_{0\{t\}} = \sum_{t \in S} \chi_{\{0\}} = \delta_0,$$

which takes the "value" $|S| = \infty$ at 0. Less extreme cases can also fail to be defined along the entire domain S. Examples are depicted in Figure 6.3. There are a few ways this situation might be treated.

(i) We could include, into the scope of discussion, unbounded functions and functions that possibly take the value ∞. This makes the $*$-“action” a mapping $S \times l^\infty(S) \to \mathbb{C}^S_\infty$. This approach is inclusive of degenerate cases such as δ_0, but merely pushes problems relating to singularities into a more complicated place. Additionally this approach does not address those $s * f$ which fail to be well-defined, but could still be argued to be bounded.

(ii) We could replace $*$ with an operator that carefully avoids singularities, and prefer to deal with a subset of $\ell^\infty(S)$ that avoids non-convergent summations. A later section in this chapter will introduce an operator $\circledast$, that is defined on the subset of non-negative functions from $\ell^\infty(S)$, and where $\circledast$ is similar enough to $*$ to be interesting.

(iii) We could regard convolution as inducing a partial action—simply accept that there will be cases where it is ill-defined, and keep to regions where it is well-defined. Since we wish to apply it to means in $\ell^\infty(S)^*$, we must also keep to the cases that are bounded. It is conceivable that $s*f$ exists in $\mathbb{C}^S$ but is unbounded, for instance, s collapses steadily increasing numbers of elements together, but never infinitely many. It is also conceivable that $s*f$ is not well-defined because of a failure to converge, but is arguably bounded.

Now, $s * f$ is well-defined and bounded exactly when $s * f \in \ell^\infty(S)$. Since S is associative, with $\ell^\infty(S)$ as a set of objects, S induces a set of arrows A_S, where for each $s \in S$ there is an arrow from each f to $s * f$ wherever $s * f \in \ell^\infty(S)$, so $(\ell^\infty(S), A_S)$ defines a semi-category. If S has an identity, then it is a category.

This last point seems interesting, not least because partial actions on C^*-algebras are the subject of current research. For our purpose here, we must ask under what conditions is $s * f$ bounded, if not $f \in \ell^1(S)$?

Lemma 6.1 If s acts injectively on the left on $\mathrm{supp}(f)$, then $s * f \in \ell^\infty(S)$.

Proof By hypothesis, $\{s * f\}(t)$ is equal to $f(x)$ for some $x \in S$ ($sx = t$) or zero (no such x). This is true for any $t \in \mathrm{supp}(f)$, and thus $s * f \in \ell^\infty(S)$. $\square$

In particular, $s * f$ exists and is bounded whenever S is left cancellative (e.g. is a group). For a semigroup generally, however, the converse does not hold: there may be $f \in \ell^\infty(S)$ such that $s * f \in \ell^\infty(S)$ but s is not injective on the support. For example, $f \in \ell^1(\mathbb{N}^0)$ given by $f(n) = 2^{-n}$, then $0 * f = \chi_{\{0\}}$.

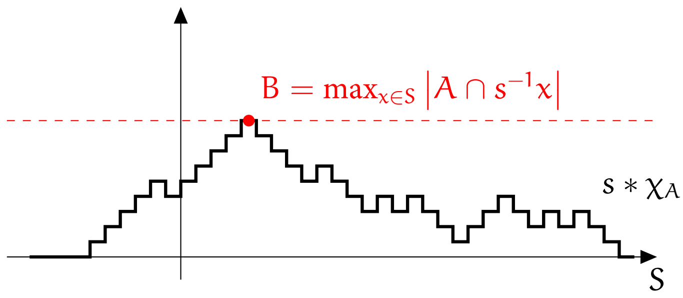

Figure 6.4: Diagram accompanying Lemma 6.2.

It is possible to be far more precise than Lemma 6.1 in characterising the elements for which $s * f$ is bounded, but it is in essence a restatement of the definition of $\|\cdot\|_\infty$. For each $s \in S$, those functions f can be thought of as

(i) behaving like elements of $\ell^1(S)$ on the subsets of S on which s does not act injectively, and

(ii) behaving like elements of $\ell^\infty(S)$ on the subsets of S on which s does act injectively.

Since these subsets of S only change with respect to s, this suggests, for each $s \in S$, a space[1] $\ell^{s*}(S)$ given by

$$f \in \ell^{s*}(S) \Leftrightarrow s * f \in \ell^\infty(S).$$

Clearly, $\ell^1(S) \subseteq \ell^{s*}(S) \subseteq \ell^\infty(S)$. By definition, $s * f \in \ell^\infty(S)$ precisely when there is some fixed finite bound $B \geq |\{s * f\}(x)| = \left|\sum_{t \in s^{-1}x} f(t)\right|$ for all $x \in S$.

Whether or not $s * f \in \ell^\infty(S)$, if $\{s * f\}(x)$ is not defined for some $x \in S$, then certainly $x \notin \mathrm{supp}(s * f)$, i.e. $\mathrm{supp}(s * f)$ can be considered well-defined even if $s * f$ is not. Therefore for all s, f, $\mathrm{supp}(s * f) = s \cdot \mathrm{supp}(f) \subseteq sS$. Therefore $s * \ell^{s*}(S)$ can be identified with a subset of $\ell^\infty(sS)$. Since every $f \in \ell^\infty(sS)$, is attainable as some $s * g$ for $g \in \ell^{s*}(S)$, it follows that

$$s * \ell^{s*}(S) \equiv \ell^\infty(sS);$$

in particular, $s * f \in \ell^\infty(S)$ if, and only if, $(s * f)|_{sS} \in \ell^\infty(sS)$.

[1]Not to be confused with either $\ell^p(S)$ or $\ell(S)^*$.

Lemma 6.2 For all $s \in S$ and $A \subseteq S$, the following conditions are equivalent.

(i) $s * \chi_A \in \ell^\infty(S)$.

(ii) There exists a finite partition $\{A_i\}_{i \in I}$ of A such that s acts injectively on the left of each A_i.

(iii) $s * \chi_A$ is simple.

Proof

(i) ⇒ (ii): Suppose $\|s * \chi_A\|_\infty = B < \infty$. B is a non-negative integer which $s * \chi_A$ attains, since the value at each point is a sum of values in $\{0, 1\}$. For all $x \in S$ we have

$$\begin{aligned} \{s * \chi_A\}(x) &= \sum_{t \in s^{-1}x} \chi_A(t) \\ &= \left|A \cap s^{-1}x\right| \\ &\leq B \quad \text{by hypothesis.} \end{aligned}$$

For $i = 1, \ldots, B$ let A_i consist of one choice element from each $(A \cap s^{-1}x) \setminus \bigcup_{j<i} A_j$ for $x \in S$ (where it is not empty).[2] Then B choices are made, each $A \cap s^{-1}x$ is exhausted, and $I = \{1, \ldots, B\}$ is finite. The finite collection $\{A_i\}_{i \in I}$ is a partition of A, since the sets of the form $s^{-1}x$ for each $x \in S$ are either empty, or distinct θ_s-equivalence classes. s acts injectively on the left of each A_i, as $A_i \cap s^{-1}x$ is either empty or a singleton set.

(ii) ⇒ (iii): Suppose there is a finite partition $\{A_i\}_{i \in I}$ of A such that s acts injectively on the left of A_i. Then $\chi_A = \sum_{i \in I} \chi_{A_i}$ and $s * \chi_{A_i} = \chi_{sA_i}$ for each $i \in I$, and thus

$$s * \chi_A = s * \sum_{i \in I} \chi_{A_i} = \sum_{i \in I} s * \chi_{A_i} = \sum_{i \in I} \chi_{sA_i},$$

which is a linear combination of finitely-many indicator functions, i.e. is simple.

(iii) ⇒ (i): If $s * \chi_A$ is simple then by definition it consists of a linear combination of finitely-many indicator functions, and thus attains some finite bound. □

[2] The Axiom of Choice is not required because the set is finite.

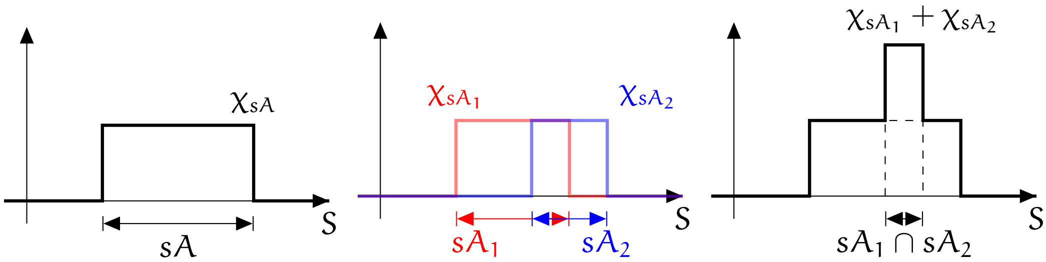

Figure 6.5: An example of $s * \chi_A \geq \chi_{sA}$. $\chi_{sA} \leq \chi_{sA_1} + \chi_{sA_2}$, where $A = A_1 \dot\cup A_2$ and s acts injectively on A_1 and A_2 but not A as a whole.

Another impediment to deducing standard results includes the difficulty in working with even simple functions. Suppose $f \in \ell^\infty(S)$ is simple, and thus there exists a finite index set I, and collections of numbers $\{a_i \in \mathbb{C} : i \in I\}$ and sets $\{A_i \in \mathcal{P}(S) : i \in I\}$ such that $f = \sum_{i\in I} a_i \chi_{A_i}$. Where it exists, $*$ distributes over $+$, and clearly if $s * f$ is bounded then $s * \chi_{A_i}$ is also bounded for each $i \in I$. Therefore,

$$s * f = \sum_{i\in I} a_i \cdot (s * \chi_{A_i}).$$

However, if the action of s is *not* injective on each A_i, it isn't at all likely that

$$\sum_{i\in I} a_i \cdot (s * \chi_{A_i}) \geq \sum_{i\in I} a_i \chi_{sA_i}$$

is saturated, and in fact $\sum_{i\in I} a_i \chi_{sA_i}$ could vary depending upon the selection of $\{A_i\}_{i\in I}$.

Fortunately, if $s * f$ is bounded then each $s * \chi_{A_i}$ is bounded and therefore by Lemma 6.2 is simple, and also, there exists a finite partition $\{B_{ij}\}_{j\in J_i}$ of each A_i such that s acts injectively on the left of each B_{ij}. Thus

$$s * f = \sum_{i\in I} a_i \sum_{j\in J_i} \chi_{sB_{ij}},$$

and hence if f is simple then so is $s * f$ (where it exists and is bounded).

6.2 Integrating $s * f$

Definition 6.3 Let $m \in \ell^\infty(S)^*$. m is *left $*$-invariant* if

$$m(f) = m(s * f)$$

for all $s \in S$ and $f \in \ell^\infty(S)$ wherever $s * f \in \ell^\infty(S)$.

The purpose of this section will be to show that Definition 6.3 is equivalent to left fair amenability of S, i.e. the existence of a left $*$-invariant (where bounded) mean is equivalent to the existence of a left fairly-invariant probability measure.

Suppose S supports a left $*$-invariant mean m as described in Definition 6.3. It is easy to see why Definition 6.3 is at least as strong as left fair amenability: when s acts injectively on the left of A, $s * \chi_A = \chi_{sA}$, and so a $*$-invariant mean can be applied to the indicator functions. To show the converse, I shall integrate with respect to μ.

First, let us consider indicator functions.

Lemma 6.4 Let S be a left fairly amenable semigroup with measure μ, $s \in S$, and $A \subseteq S$. If $s * \chi_A \in \ell^\infty(S)$ then

$$\int (s * \chi_A)\, d\mu = \int \chi_A d\mu.$$

Proof If $s * \chi_A \in \ell^\infty(S)$, then there is the finite partition $\{A_i\}_{i \in I}$ of A provided by Lemma 6.2 such that s acts injectively on the left of each A_i. Then

$$\begin{aligned} \int (s * \chi_A)\, d\mu &= \int \left(\sum_{i \in I} \chi_{sA_i} \right) d\mu \\ &= \sum_{i \in I} \mu(sA_i) \quad \text{by definition} \\ &= \sum_{i \in I} \mu(A_i) \quad \because \text{fair invariance} \\ &= \mu(A) \quad \because \text{finitely additive} \\ &= \int \chi_A d\mu \quad \text{again by definition,} \end{aligned}$$

as required. □

Lemma 6.5 Let S be a left fairly amenable semigroup with measure μ, $s \in S$, and $f \in \ell^\infty_+(S)$ is a simple function. If $s * f \in \ell^\infty(S)$ then

$$\int (s * f)\, d\mu = \int f d\mu.$$

Proof There are the requisite finite index set I, sets $\{A_i\}_{i\in I}$ and values $a_i \in \mathbb{R}^+$ for $i \in I$ such that $f = \sum_{i\in I} a_i \chi_{A_i}$. If $s * f \in \ell^\infty(S)$ then $s * \chi_{A_i} \in \ell^\infty(S)$ for each $i \in I$, and therefore

$$\begin{aligned}
\int (s * f)\, d\mu &= \int \left(s * \sum_{i\in I} a_i \chi_{A_i} \right) d\mu \\
&= \int \left(\sum_{i\in I} a_i \cdot (s * \chi_{A_i}) \right) d\mu \\
&= \sum_{i\in I} a_i \int (s * \chi_{A_i})\, d\mu \\
&= \sum_{i\in I} a_i \int \chi_{A_i} d\mu \qquad \because \text{Lemma 6.4} \\
&= \int \left(\sum_{i\in I} a_i \chi_{A_i} \right) d\mu \\
&= \int f d\mu
\end{aligned}$$

as required. □

Let $\ell^\infty_+(S)$ denote the subset of $\ell^\infty(S)$ consisting of bounded real-valued non-negative functions on S.

Not every simple function $h \leq s * f$ is of the form $s * g$ for a simple $g \in \ell^\infty_+(S), g \leq f$. For example, let $f \in \ell^1(\mathbb{N}^0)$ with $f(n) = 1/2^n$ for each n. Then $\{0 * f\}(0) = 2$. $h = 0 * f$ itself is simple. However, there is no $g \leq f$ such that g is simple and $0 * g = h$ (that would require either g to be non-simple or $g > f$). It is nevertheless sufficient to range over all functions of the form $s * g$ for simple $g \leq f$, when integrating $s * f$, as $s * g$ approximates $s * f$ increasingly well as g gains detail.

Lemma 6.6 Let S be a left fairly amenable semigroup with measure μ, $s \in S$, and $f \in \ell^\infty_+(S)$. If $s * f \in \ell^\infty_+(S)$ then

$$\int (s * f)\, d\mu = \int f d\mu.$$

Proof If $s * f \in \ell^\infty_+(S)$ then $s * h \in \ell^\infty_+(S)$ for every simple function $h \leq f$. Thus

$$\begin{aligned}\int (s * f)\, d\mu &= \sup\left\{\int h d\mu : h \leq (s * f), h \text{ is simple}\right\} \quad \text{by definition}\\ &= \sup\left\{\int (s * g) d\mu : (s * g) \leq (s * f), g \text{ is simple}\right\}\\ &= \sup\left\{\int g d\mu : g \leq f, g \text{ is simple}\right\} \quad \because \text{Lemma 6.5}\\ &= \int f d\mu \quad \text{by definition}\end{aligned}$$

as required. □

The next lemma is is entirely routine.

Lemma 6.7 Let S be a left fairly amenable semigroup with measure μ, $s \in S$, and real-valued $f \in \ell^\infty(S)$. If $s * f \in \ell^\infty(S)$ then

$$\int (s * f)\, d\mu = \int f d\mu.$$

Proof There exist $f^+, f^- \in \ell^\infty_+(S)$ such that $f = f^+ - f^-$. If $s * f \in \ell^\infty(S)$ then so too $s * f^+$ and $s * f^-$, thus

$$\begin{aligned}\int (s * f) d\mu &= \int (s * (f^+ - f^-)) d\mu\\ &= \int ((s * f^+) - (s * f^-)) d\mu\\ &= \int (s * f^+) d\mu - \int (s * f^-) d\mu\\ &= \int f^+ d\mu - \int f^- d\mu \quad \because \text{Lemma 6.6}\\ &= \int (f^+ - f^-) d\mu\\ &= \int f d\mu\end{aligned}$$

as required. □

The complex-valued case is even more pedestrian, so it is omitted.

Theorem 6.8 (Main Theorem) A semigroup S is left fairly amenable if, and only if, there exists a left $*$-invariant mean in $\ell^\infty(S)^*$.

Proof Suppose S is left fairly amenable with the finitely-additive measure μ. By Lemma 6.7, the integral with respect to μ is $*$-invariant, therefore use the mean $m \in \ell^\infty(S)^*$ given by setting

$$m(f) := \int f d\mu \quad \text{for all } f \in \ell^\infty(S).$$

Conversely, if S supports a left $*$-invariant mean m, define the measure $\mu \in [0,1]^{\mathcal{P}(S)}$ by setting

$$\mu(A) := m(\chi_A) \quad \text{for all } A \in \mathcal{P}(S).$$

Then, if $s \in S$ acts injectively on the left of $A \in \mathcal{P}(S)$, $s * \chi_A = \chi_{sA}$, and then

$$\begin{aligned} \mu(sA) &= m(\chi_{sA}) \\ &= m(s * \chi_A) \\ &= m(\chi_A) = \mu(A), \end{aligned}$$

as required. □

6.3 Modified convolution

Let us return to the notion that $*$ is often ill-defined on $\ell^\infty(S)$. For any S, $*$ is in general only guaranteed to be defined on $\ell^1(S)$, and, if we are fortunate, is defined on some superset of $\ell^1(S)$. An expression like $0 * \chi_S$ is certainly undesirable.

Another possible solution to this problem is to introduce a different action, one that is induced by a less problematic operator. The operator $\circledast$ introduced below is well-defined for all non-negative bounded functions, and has the property that $s \circledast \chi_A = \chi_{sA}$ for all $s \in S$ and $A \subseteq S$.

Let us for now restrict our attention to bounded, real-valued, non-negative functions in $\ell^\infty(S)$, here denoted $\ell^\infty_+(S)$. $\ell^\infty_+(S)$ lacks additive inverses, but is otherwise very similar to $\ell^\infty(S)$. Thus it is a semi-vector space (Janyska, Modugno, and Vitolo, 2007), that is complete under the ordinary $\ell^\infty(S)$-norm, so might be termed a *Banach semi-(vector) space*.

Recall, for any semigroup S, the convolution operation $*$ is defined in the usual

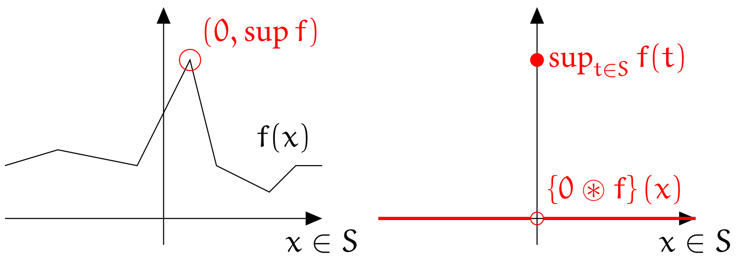

Figure 6.6: The result of the left $\circledast$-action of 0 on some $f \in \ell_+^\infty(S)$.

manner by setting

$$\{f * g\}(x) := \sum_{st=x} f(s)\,g(t) \quad \text{for all } f, g \in \ell^\infty(S), x \in S,$$

but only wherever the summation converges for all $x \in S$. Otherwise, the operation remains undefined. Note that the inner summation is taken over all pairs s, t such that $st = x$, which is itself the key concept. $\sum$ could conceivably be replaced with any combining operation.

Definition 6.9 Define the operation $\circledast$ on $\ell_+^\infty(S)$ by setting[3]

$$\{f \circledast g\}(x) := \sup_{st=x} f(s)\,g(t)$$

for all $f, g \in \ell_+^\infty(S)$ and $x \in S$. Note that unlike $*$, $\circledast$ remains well-defined for all f, g. $\circledast$ thus induces a left action of $s \in S$ on $\ell_+^\infty(S)$, written $s \circledast f$, given by

$$(s \circledast f)(x) = \sup_{st=x} f(t) \quad \text{for all } s, x \in S, f \in \ell_+^\infty(S).$$

Additionally, define the operation $\vee$ on $\ell_+^\infty(S)$ by setting

$$\{f \vee g\}(x) := \max\{f(x), g(x)\}$$

for all $f, g \in \ell_+^\infty(S)$ and $x \in S$.

[3] Don't forget that $x \in S$ is identified with $\chi_{\{x\}}$.

Remark 6.10 A simple function $f = \sum_{i \in I} a_i \chi_{A_i}$ can be re-expressed as

$$f = \bigvee_{i=1}^{n} a_i \chi_{A_i}$$

assuming that the $\{A_i\}_{i \in I}$ are pairwise disjoint.

Lemma 6.11 The following results are clear. Let S be a semigroup and $f \in \ell_+^\infty(S)$.

(i) For all $s \in S$,

$$s \circledast t = \chi_{\{s\}} \circledast \chi_{\{t\}} = \chi_{\{st\}} = st,$$

and thus the map $s \mapsto \chi_{\{s\}}$ is an injective semigroup homomorphism.

(ii) For any $s \in S$, and $A \subseteq S$,

$$\begin{aligned} s \circledast \chi_A &= \sum_{u \in S} \left(\sup_{t \in s^{-1}\{u\}} \chi_A(t) \right) u \\ &= \sum_{u \in S} \left(\begin{cases} 1 & \text{if } \exists t \in A : st = u \\ 0 & \text{otherwise} \end{cases} \right) u \\ &= \chi_{sA}, \end{aligned}$$

and similarly on the right, so $\circledast$ preserves the regular left and right actions of elements on sets.

(iii) It is also clear that $\chi_A \circledast \chi_B = \chi_{AB}$, for any sets $A, B \subseteq S$. (Compare with ordinary convolution.)

(iv) The action on f can be thought of as an action on the support of f, i.e.

$$\operatorname{supp}(s \circledast f) = s \cdot \operatorname{supp}(f); \quad \operatorname{supp}(f \circledast s) = \operatorname{supp}(f) \cdot s.$$

(v) If G is a group and $g \in G$, then

$$g \circledast \phi = g * \phi = g^{-1} \cdot \phi \quad \text{for all } \phi \in \ell_+^\infty(G).$$

Lemma 6.12 Let S be a semigroup and $f \in \ell_+^\infty(S)$. If s acts injectively on the left of $\operatorname{supp}(f)$, then

$$(s \circledast f)(st) = (s * f)(st) = f(t) \quad \text{for all } t \in S.$$

In particular, if s acts injectively on the left of A, then $s * \chi_A = s \circledast \chi_A = \chi_{sA}$.

Proof For all $a, b \in \operatorname{supp}(f)$, $sa = sb \Rightarrow a = b$ and therefore

$$\begin{aligned}(s \circledast f)(st) &= \sup_{sx=st} f(x) \\ &= \sup_{x \in s^{-1}\{st\} \cap \operatorname{supp}(f)} f(x) \\ &= f(t)\end{aligned}$$

for all $t \in S$. □

Thus $\circledast$ is equivalent to $*$ in cases where the action is injective.

Lemma 6.13 For any semigroup S:

(i) $\circledast$ is indeed a binary operator: if $f, g \in \ell_+^\infty(S)$ then $f \circledast g \in \ell_+^\infty(S)$, with $\|f\|_\infty \|g\|_\infty \geq \|f \circledast g\|_\infty$.

(ii) $\circledast$ is associative for $\ell_+^\infty(S)$, and distributes over $\vee$.

(iii) If $\lambda \geq 0$ then multiplication by λ over $\circledast$ is bilinear, i.e. $(\lambda f) \circledast g = f \circledast (\lambda g) = \lambda(f \circledast g)$.

Proof

(i) For all $s, t \in S$, we have $|f(s)\, g(t)| = |f(s)|\,|g(t)| \leq \|f\|_\infty \|g\|_\infty$. Thus $\|f\|_\infty \|g\|_\infty$ is a bound on $f \circledast g$ (not necessarily the best possible such bound), and immediately, $\|f\|_\infty \|g\|_\infty \geq \|f \circledast g\|_\infty$.

(ii) Note that if $k \geq 0$ then for any set of real numbers A, $k \sup A = \sup kA$. If

$f, g, h \in \ell_+^\infty(S)$, and $u \in S$,

$$\begin{aligned}((f \circledast g) \circledast h)(u) &= \sup_{st=u}(f \circledast g)(s)\,h(t) \\ &= \sup_{st=u}\left(\sup_{vw=s} f(v)\,g(w)\right) h(t) \\ &= \sup_{vwt=u} f(v)\,g(w)\,h(t) \\ &= \sup_{st=u} f(s)\left(\sup_{vw=t} g(v)\,h(w)\right) \\ &= \sup_{st=u} f(s)\,(g \circledast h)(t) \\ &= (f \circledast (g \circledast h))(u),\end{aligned}$$

thus, in particular, $\circledast$ is associative on $\ell_+^\infty(S)$.

Also,

$$\begin{aligned}[f \circledast (g \vee h)](u) &= \sup_{st=u} f(s)\,(g \vee h)(t) \\ &= \sup_{st=u} \max\{f(s)\,g(t), f(s)\,h(t)\} \\ &= \max\left\{\sup_{st=u} f(s)\,g(t), \sup_{st=u} f(s)\,h(t)\right\} \\ &= [(f \circledast g) \vee (f \circledast h)](u)\end{aligned}$$

and similarly on the right.

(iii) If $\lambda \geq 0$ then

$$\begin{aligned}(\lambda \cdot (f \circledast g))(u) &= \sup_{st=u} \lambda f(s)\,g(t) \\ &= ((\lambda f) \circledast g)(u) = (f \circledast (\lambda g))(u)\end{aligned}$$

so $\circledast$ is bilinear for non-negative scalars. □

Remark 6.14 The space $(\ell_+^\infty(S), \vee, \circledast)$ is not only a complete semi-vector space but also an idempotent semiring similar to the max-plus dioid (Gunawardena, 2001), and so it might tentatively be called a *Banach semi-algebra*.

Remark 6.15 Some reasons that we have to be restricted to non-negative functions and scalars are as follows. Firstly, bilinearity with negative scalars does not work. Suppose $0 \in S$ and the function $f \in \ell^\infty(S)$ (not necessarily non-negative) has a maximum of M and a minimum of m, with $m \neq M$. We want, for example, $-(0 \circledast f) = 0 \circledast (-f)$, but we don't get it as a consequence of applying the definition directly:

$$-(0 \circledast f) = -\sum_{u \in S} \left(\sup_{0t=u} f(t) \right) \chi_{\{u\}} = -\left(\sup_{t \in S} f(t) \right) \chi_{\{0\}} = -M\chi_{\{0\}}$$

whereas

$$0 \circledast (-f) = \left(\sup_{t \in S} -f(t) \right) \chi_{\{0\}} = -m\chi_{\{0\}}.$$

It would also be nice if $\circledast$ distributed over addition, even for only non-negative functions. Alas, while

$$\begin{aligned} (f \circledast (g+h))(u) &= \sup_{st=u} f(s)(g+h)(t) \\ &= \sup_{st=u} [f(s) g(t) + f(s) h(t)] \end{aligned}$$

is straightforward, it is conceivable that

$$\sup_{st=u} [f(s) g(t) + f(s) h(t)] < \left(\sup_{st=u} f(s) g(t) + \sup_{st=u} f(s) h(t) \right)$$

strictly. For example, suppose a, b and c, d are the only pairs for which $ab = u = cd$, with $f(a) = 1, f(c) = 2$, $g(b) = 2$, $g(d) = 0$, and $h = 1$. It frequently won't work even for the $\circledast$-action of an element on the sum of two functions with disjoint support: for example, suppose f and g are disjoint, but $s^{-1}\{u\}$ intersects both supports for some u. So it seems that there's no hope that $\circledast$ could make $\ell^\infty(S)$ into an algebra this way.

However, it is possible to carefully extend $\circledast$ so that the left semigroup action remains well-defined on all $\ell^\infty(S)$.

Definition 6.16 For all $\phi \in \ell^\infty(S)$, if ϕ is real-valued, then there are disjoint $\phi^+, \phi^- \in \ell^\infty_+(S)$ such that $\phi = \phi^+ - \phi^-$. Define the left $\circledast$-action of s on ϕ, $s \circledast \phi$, by

$$s \circledast \phi := (s \circledast \phi^+) - (s \circledast \phi^-) \quad \text{for all } s \in S.$$

If ϕ is complex-valued, there are real-valued $\phi_R, \phi_I \in \ell^\infty(S)$ such that $\phi = \phi_R + i\phi_I$, so, define (reusing the above)

$$s \circledast \phi := (s \circledast \phi_R) + i(s \circledast \phi_I) \quad \text{for all } s \in S.$$

Define the right $\circledast$-action of each $s \in S$ on ϕ, $\phi \circledast s$, similarly. Thus the semigroup-action part of $\circledast$ is extended to all $\ell^\infty(S)$ in a way that is at least well-defined.

Finally, note that while the ordinary left dual and right dual actions on $\ell^1_+(S)$ don't *quite* correspond under the ordinary duality to the $\circledast$ actions on $\ell^\infty_+(S)$, there is this. Let $\langle \cdot, \cdot \rangle_{\sup}$ denote the operation given by

$$\langle f, g \rangle_{\sup} = \sup_{t \in S} f(t)\, g(t),$$

and is defined wherever the right-hand side is well-defined.

Lemma 6.17 For each $\phi \in \ell^\infty_+(S)$ let $\phi\check{} \in \ell^1_+(S)^*$ be given by

$$\phi\check{}(f) := \sup_{t \in S} f(t)\, \phi(t)$$

for all $f \in \ell^1_+(S)$. Then the left $\circledast$ action on $\ell^\infty_+(S)$ is "sup-dual" to the right dual action on $\ell^1_+(S)$ in the sense that

$$\phi\check{}(f \cdot s) = (s \circledast \phi)\check{}(f), \quad \text{or equivalently,} \quad \langle \phi, f \cdot s \rangle_{\sup} = \langle s \circledast \phi, f \rangle_{\sup}$$

for all $f \in \ell^1_+(S)$ and $s \in S$. (Compare with Lemmas 2.9, 4.4, 7.17.)

Proof By definition,

$$\begin{aligned}
(s \circledast \phi)\check{}(f) &= \sup_{t \in S} f(t)\, (s \circledast \phi)(t) \\
&= \sup_{t \in S} f(t) \left(\sup_{u \in s^{-1}\{t\}} \phi(u) \right) \\
&= \sup_{t \in S} \left(\sup_{u \in s^{-1}\{t\}} f(su)\, \phi(u) \right) \\
&= \sup_{t \in S} f(st)\, \phi(t) \\
&= \phi\check{}(fs),
\end{aligned}$$

for all ϕ, f, s as appropriate. □

6.4 $\circledast$-invariant means

Using $\circledast$ the following definitions are inspired by the fair amenability concept from Chapter 5:

Definition 6.18 Let $m \in \ell^\infty(S)^*$.

(i) m is *left sub-$\circledast$-invariant* if $|m(f)| \geq |m(s \circledast f)|$ for all $s \in S$ and $f \in \ell^\infty(S)$.

(ii) m is *left fairly $\circledast$-invariant* if, whenever s acts injectively on $\mathrm{supp}(f)$, we have $m(f) = m(s \circledast f)$.

The purpose of this section is to prove that a semigroup S is left [right] fairly amenable if, and only if, there exists a left [right] fairly $\circledast$-invariant mean on $\ell^\infty(S)$. If s is injective on the left of A, then $s \circledast \chi_A = \chi_{sA}$, and so if a semigroup S supports a left fairly $\circledast$-invariant mean, it is left fairly amenable. As before, showing the converse involves integrating with respect to a left fairly invariant μ. Note that, to find variants of this condition, $\circledast$ may be replaced by any other operator inducing an action of S.

Lemma 6.19 For a semigroup S, the measure μ is left fairly invariant if, and only if, $\int d\mu$ is left fairly $\circledast$-invariant for the indicator functions. μ is left sub-invariant if, and only if, $\int d\mu$ is left sub-$\circledast$-invariant.

Proof If μ is left sub-invariant, then

$$\begin{aligned}\int \chi_A d\mu &= \mu(A) \\ &\geq \mu(sA) \quad \because \text{ sub-invariance} \\ &= \int \chi_{sA} d\mu \\ &= \int (s \circledast \chi_A)\, d\mu \quad \text{by Lemma 6.11.}\end{aligned}$$

If μ is left fairly invariant and s acts injectively on A, then equality holds. The converse is analogous. □

Lemma 6.20 Suppose $f \in \ell_+^\infty(S)$ is simple. If μ is a left sub-invariant finitely-additive measure on S, then

$$\int f d\mu \geq \int (s \circledast f)\, d\mu.$$

If μ is left fairly invariant and s acts injectively on the left of $\mathrm{supp}(f)$, then the inequality is saturated.

Proof By hypothesis, f is simple, so let $f = \sum_{i=1}^n a_i \chi_{A_i}$ for the finite pairwise disjoint collection $\{A_i\}_{i=1}^n$ as in Definition 1.5.

1. $s \circledast f = \bigvee_{i=1}^n a_i \chi_{sA_i}$.
   PROOF: See Lemma 6.13.
2. $\int f d\mu = \sum_{i=1}^n a_i \mu(A_i)$.
   PROOF: From linearity of integration,
   $$\begin{aligned}\int f d\mu &= \int \left(\sum_{i=1}^n a_i \chi_{A_i}\right) d\mu \\ &= \sum_{i=1}^n a_i \int \chi_{A_i} d\mu \\ &= \sum_{i=1}^n a_i \mu(A_i).\end{aligned}$$
3. If μ is sub-invariant, then $\int f d\mu \geq \int (s \circledast f)\, d\mu$.
   The subinvariant case can be proved with induction. We can see that for any n-step simple function f, we have the partial sums
   $$f_k = \sum_{i=1}^k a_i \chi_{A_i},$$
   for $k = 0, \ldots, n$, with f_k being a k-step function, and obviously $f = f_n$.
   3.1. Base case: For $f = 0$ (a 0-step function), $\int f d\mu \geq \int (s \circledast f)\, d\mu$, as both sides take the value 0.
   3.2. If, for a given $(n+1)$-step function $f = f_{n+1}$, we have $\int f_n d\mu \geq \int (s \circledast f_n)\, d\mu$, then $\int f d\mu \geq \int (s \circledast f)\, d\mu$.

PROOF: Either sA_{n+1} is disjoint from $\operatorname{supp}(s \circledast f_n)$ or it is not. If it is disjoint,

$$\begin{aligned}\int (s \circledast f_{n+1})\, d\mu &= \int (s \circledast f_n)\, d\mu + a_i \mu\,(sA_{n+1}) \\ &\leq \int f_n d\mu + a_i \mu\,(A_{n+1}) \\ &= \int f_{n+1} d\mu.\end{aligned}$$

Otherwise, sA_{n+1} will intersect possibly all the other sets, and so

$$\int (s \circledast f_{n+1})\, d\mu = \int (s \circledast f_n)\, d\mu + a_i \mu\,(sA_{n+1}) - \epsilon,$$

where $\epsilon \geq 0$ is some amount accounting for the overlap, and thus the desired inequality holds again.

Hence, by induction, left sub-invariance of μ implies $\int f d\mu \geq \int (s \circledast f)\, d\mu$ for any non-negative simple f.

If s is injective on $\operatorname{supp}(f)$, then each sA_i will necessarily be disjoint from the others. Thus if μ is left fairly invariant we can integrate simple functions directly to see that

$$\begin{aligned}\int f d\mu &= \sum_{i=1}^{n} a_i \mu\,(A_i) \quad \text{by definition} \\ &= \sum_{i=1}^{n} a_i \mu\,(sA_i) \\ &= \int (s \circledast f)\, d\mu,\end{aligned}$$

thus completing the lemma. □

Remark 6.21 Given a linear functional $m \in \ell_+^\infty(S)^*$ defined on simple functions, m is extended to all $\ell_+^\infty(S)$ by setting

$$m(f) := \sup\{m\,(g) : g \leq f \text{ and } g \in \ell_+^\infty(S) \text{ is simple}\} \quad \text{for all } f \in \ell_+^\infty(S).$$

With the case of $s * f$ for non-negative but not necessarily simple f, not every simple $h \leq s * f$ arises as some $s * g$, but it was enough that the $s * g$ for simple $g \leq f$ could be used to integrate $s * f$. With $s \circledast f$, there is a stronger result, shown in the next lemma.

Lemma 6.22 If $h \in \ell_+^\infty(S)$ is a simple function and $h \leq s \circledast f$, then there exists a simple $g \in \ell_+^\infty(S)$ such that $g \leq f$ and $h = s \circledast g$.

Proof

1. h is simple, so by definition there exists $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$ and pairwise disjoint

$A_1, A_2, \ldots, A_n \subseteq S$, such that

$$h = \sum_{i=1}^{n} a_i \chi_{A_i}.$$

2. Since $h \leq s \circledast f$, $\mathrm{supp}(h) \subseteq \mathrm{supp}(s \circledast f) = s \cdot \mathrm{supp}(f)$.
3. Each $A_i = sB_i$ for some $B_i \subseteq \mathrm{supp}(f)$, also satisfying $a_i \chi_{B_i} \leq f$.

   PROOF:

   3.1. For each $i \in I$ and $x \in A_i$ let $C_x := \{t : st = x, f(t) \geq a_i\}$.

   3.2. C_x is nonempty.

   PROOF: If C_x is empty then either there is no t such that $st = x$, contradicting $x \notin A_i \subseteq s \cdot \mathrm{supp}(f)$, otherwise, $f(t) < a_i$ for every t, in which case $(s \circledast f)(x) < a_i = h(x)$, contradicting $h \leq s \circledast f$.

   3.3. For each $i \in I$, construct the set B_i by include any or all elements of C_x for all $x \in A_i$.
4. Thus after obtaining such a finite collection of sets $B_1, B_2, \ldots B_n$, we can construct g as follows:

$$g = \sum_{i=1}^{n} a_i \chi_{B_i},$$

and thus $h = s \circledast g$ and $0 \leq g \leq f$, as desired. □

Lemma 6.23 Given $f \in \ell_+^\infty(S)$ (not necessarily simple), if μ is a left sub-invariant finitely-additive measure on S, then $\int f d\mu \geq \int (s \circledast f)\, d\mu$. If μ is left fairly invariant and s acts injectively on the left of $\mathrm{supp}(f)$, then $\int f d\mu = \int (s \circledast f)\, d\mu$.

Proof By Lemma 6.22, ranging over all non-negative functions of the form $s \circledast g$ for simple g where $g \leq f$ will suffice to include all non-negative simple functions less than $s \circledast f$. For left sub-invariant μ, we then have for each simple $g \leq f$ that $m_\mu(g) \geq m_\mu(s \circledast g)$, and so

$$\begin{aligned}
\int f d\mu &= \sup\left\{\int g d\mu : g \leq f, g \text{ is simple}\right\} \\
&\geq \sup\left\{\int (s \circledast g)\, d\mu : (s \circledast g) \leq (s \circledast f), (s \circledast g) \text{ is simple}\right\} \\
&= \sup\left\{\int g d\mu : g \leq (s \circledast f), g \text{ is simple}\right\} \\
&= \int (s \circledast f)\, d\mu.
\end{aligned}$$

If s is injective on the left of $\mathrm{supp}(f)$, then it is injective on each $\mathrm{supp}(g) \subseteq$

$\mathrm{supp}(f)$, so for fairly invariant μ, $\int g d\mu = \int (s \circledast g)\, d\mu$ for all non-negative simple $g \leq f$, and the inequality above is saturated, so that $\int f d\mu = \int (s \circledast f)\, d\mu$. $\square$

Theorem 6.24 A semigroup S is left fairly amenable if, and only if, there exists a left fairly $\circledast$-invariant mean in $\ell^\infty(S)^*$.

Proof Lemma 6.19 gives a suitable finitely-additive measure given such a mean.

Conversely, for every real-valued $f \in \ell^\infty(S)$, there exists $f^+, f^- \in \ell^\infty_+(S)$ such that $f = f^+ - f^-$. Defining m_μ by

$$m_\mu(f) := \int f d\mu \quad \text{for all } f \in \ell^\infty(S),$$

if s acts on the left of $\mathrm{supp}(f)$ injectively, then

$$\begin{aligned}
m_\mu(s \circledast f) &= \int (s \circledast f)\, d\mu \\
&= \int ((s \circledast f^+) - (s \circledast f^-))\, d\mu \quad \because \text{Definition 6.16} \\
&= \int (s \circledast f^+)\, d\mu - \int (s \circledast f^-)\, d\mu \quad \because \text{linearity} \\
&= \int f^+ d\mu - \int f^- d\mu \quad \because \text{Lemma 6.23} \\
&= \int f d\mu = m_\mu(f)
\end{aligned}$$

as required. For complex-valued f, since there exists real-valued $f_R, f_I \in \ell^\infty(S)$ such that $f = f_R + i f_I$, and so similarly,

$$\begin{aligned}
m_\mu(s \circledast f) &= \int (s \circledast f)\, d\mu \\
&= \int ((s \circledast f_R) + i(s \circledast f_I))\, d\mu \quad \because \text{Definition 6.16} \\
&= \int (s \circledast f_R)\, d\mu + i \int (s \circledast f_I)\, d\mu \\
&= \int f_R d\mu + i \int f_I d\mu \quad \because \text{the above} \\
&= \int f d\mu = m_\mu(f),
\end{aligned}$$

as required. $\square$

6.5 Generalised convolution

Perhaps there is some value in generalising the process in devising the operator $\circledast$. Recall:

$$\{s * f\}(x) = \sum_{st=x} f(t).$$

The pointwise sum of functions forms the basis of the convolution partial action. But if functions are generalisations of sets, the sum is not a good generalisation of set union—for example, almost no functions are idempotent under summation. Perhaps in order to find a good action, we want a combining operation, similar to summation, perhaps corresponding to a semilattice homomorphism from $\mathcal{P}(S)$ to an operation on $\ell^\infty(S)$, that will replace the summation. The combining operation sup was such an example.

Recall the $\vee$ operator. It is dual to $\wedge$, defined by

$$\{f \wedge g\}(t) := \min\{f(t), g(t)\} \quad \text{for all } t \in S.$$

The operators $\vee$ and $\wedge$ extend naturally to bounded collections of non-negative functions, by setting

$$\left\{\bigvee_{i\in I} f_i\right\}(t) := \sup_{i\in I} f_i(t), \quad \left\{\bigwedge_{i\in I} f_i\right\}(t) := \inf_{i\in I} f_i(t).$$

It is relatively easy to see that if $\{f_i\}_{i\in I}$ is a finite collection of bounded real-valued functions, then so too are $\bigvee_{i\in I} f_i$ and $\bigwedge_{i\in I} f_i$—simply use the greatest bound. On the other hand, we can show that this does not always produce a bounded function for infinite collections by choosing an infinite collection of functions whose bounds diverge, for instance, $\{n\chi_S\}_{n=0}^{\infty}$. This can be repaired by setting a common bound for the whole collection, and hence *bounded collection*.

Now

$$\chi_{A\cup B} = \chi_A \vee \chi_B, \quad \chi_{A\cap B} = \chi_A \wedge \chi_B$$

making χ a lattice homomorphism from $\mathcal{P}(S)$ considered as a Boolean algebra.

6.5.1 Generalised semigroup convolution

The basic idea with the convolution $\circledast$ above is to use "sup" in place of "$\sum$". This has the advantage of maintaining boundedness and therefore closure in $\ell^\infty(S)$, but has some drawbacks.

For each $f, g \in \ell^\infty(S)$ we have a function P_{fg} which maps $(s,t) \mapsto f(s)g(t)$. For every $u \in S$, we effectively are asking for a number related to the values of P_{fg} over every s, t such that $st = u$: some "$\text{mean}_{st=u} f(s)g(t)$" with nice enough properties.

Example 6.25 Suppose S has the property that for each u there are at most countably many s, t such that $st = u$. If there are finitely many, we can repeat them infinitely to form a countable sequence. Thus for each $u \in S$ and $f, g \in \ell^\infty(S)$ we can fix a sequence of values $\{a_i\}_{i=1}^\infty = \{f(s)g(t)\}_{st=u}$. Then put

$$(f \bar{*} g)(u) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i.$$

Perhaps there is a more general and elegant way of setting this up. Let $\Lambda(u) := \{s : (\exists t)(st = u)\}$—the left divisors of u—and similarly $P(u) := \{t : (\exists s)(st = u)\}$—the right divisors of u. Consider

$$(f{*}g)(u) = \sum_{st=u} f(s)g(t) = \sum_{s\in\Lambda(u)} f(s) \left(\sum_{t\in s^{-1}\{u\}} g(t) \right) = \sum_{t\in P(u)} g(t) \left(\sum_{s\in\{u\}t^{-1}} f(s) \right)$$

which demonstrates the similarity to usual convolution (of real functions using integrals). Perhaps we want to simultaneously replace both summations such that the function remains bounded, but also has the desirable properties that would make $\ell^\infty(S)$ similar to an algebra. Another way to think about this is that the problem is in deciding how "$f\left(s^{-1}x\right)$" should be interpreted.

6.5.2 Restricted means and functions

For real-valued functions $f \in \ell^\infty(S)$, the map $f \mapsto \sup_{x\in S} f(s)$ is similar to a mean. It fails to be a mean as it is not linear. So, some slight modification to means might be useful as a generalisation for sup. If means generalise integrals, then "means relative to some set" would be a generalisation of integrals taken over some set. To ease matters, I shall apply an idea from three-valued logic.

For convenience, let $\mathbf{N}$ (for "null," or "not defined," or "unknown") be a new additive and multiplicative zero adjoined onto the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, i.e. for any $x \in \mathbb{K} \cup \{\mathbf{N}\}$,

$$x + \mathbf{N} = \mathbf{N} = \mathbf{N} + x, \quad x\mathbf{N} = \mathbf{N} = \mathbf{N}x.$$

Additionally, for the convenience of setting up a norm, set

$$|\mathbf{N}| = \mathbf{N} \quad \text{and} \quad \mathbf{N} \leq x.$$

Now define a modified indicator function $\hat{\chi}_A$ by setting for all t,

$$\hat{\chi}_A(t) = \begin{cases} 1 & \text{if } t \in A \\ \mathbf{N} & \text{if } t \notin A. \end{cases}$$

Using $\hat{\chi}$, we can reduce the domain $\mathbf{d}(f)$ of a function f algebraically. Let $f : S \to \mathbb{K}$ and $A \subseteq S$. The *function* f *restricted to* A, denoted $f|A$, is now the pointwise product $f \cdot \hat{\chi}_A$. Since any function $S \to \mathbb{K} \cup \{\mathbf{N}\}$ arises as such a product, we will generally write $f|A$ to imply that f is unrestricted and $(f|A)(t) = \mathbf{N}$ only when $t \notin A$, i.e. so that the following is true.

$$\mathbf{d}(f|A) := \{s : (f|A)(s) \neq \mathbf{N}\} = A.$$

Note that as for ordinary indicator functions,

$$\hat{\chi}_A \cdot \hat{\chi}_B = \hat{\chi}_{A\cap B}, \quad \text{and} \quad \operatorname{supp}(\hat{\chi}_A) = A$$

for any subsets A, B, and therefore

$$(f|A)|B = f \cdot \hat{\chi}_A \cdot \hat{\chi}_B = f|(A \cap B) = (f|B)|A.$$

Finally, note that $(f|A)|\emptyset = \mathbf{N}$ constant, for any f and A.

Definition 6.26 Let $\ell^\infty_{\mathbf{N}}(S)$ denote the space of bounded functions on S, including $\mathbf{N}$: that is, define for all $f : S \to \mathbb{K}, A \subseteq S$ the norm

$$\|f|A\|_\infty := \sup_{s\in A} |f(s)|$$

and if $A = \emptyset$ (i.e. $f|A = \mathbf{N}$), define $\|f|A\|_\infty = \mathbf{N}$. $f|A \in \ell^\infty_{\mathbf{N}}(S)$ if, and only if,

$\|f|A\|_\infty < \infty$ or $\|f|A\|_\infty = \mathbf{N}$ (in which case $f|A = \mathbf{N}$).

Lemma 6.27 As extended to the $\mathbf{N}$-inclusive space given above, $\|\cdot\|_\infty$ is actually a norm deserving of the notation, and $\ell^\infty_\mathbf{N}(S)$ is a Banach space. For any $f \in \ell^\infty(S)$ and $A \subseteq S$, $f|A \in \ell^\infty_\mathbf{N}(S)$, and every $g \in \ell^\infty_\mathbf{N}(S)$ arises this way.

Proof

1. We require $\|\alpha(f|A)\|_\infty = |\alpha|\,\|f|A\|_\infty$. Cases to be concerned about are either $\alpha = \mathbf{N}$ or $f|A = \mathbf{N}$. In either case the condition holds trivially.

2. We require $\|f|A + g|B\|_\infty \leq \|f|A\|_\infty + \|g|B\|_\infty$. Again, if either $f|A$ or $g|B$ is the constant $\mathbf{N}$, the condition holds trivially. But if $A \cap B = \emptyset$ then we could have $\|f|A + g|B\|_\infty = \mathbf{N}$ and $\|f|A\|_\infty \neq \mathbf{N} \neq \|g|B\|_\infty$. Fortunately we set $\mathbf{N} \leq x$ for all x previously. On the other hand there is no instance where $f|A + g|B \neq \mathbf{N}$ and $f|A \neq \mathbf{N} \neq g|B$.

3. For a norm we usually require $\|f|A\|_\infty = 0$ if and only if $f|A = 0$. $f|A = 0$ if and only if $f = 0$ and $A = S$. But in this case there are clearly many sets A such that $\|0|A\|_\infty = 0$. Instead, let's require $\|f|A\|_\infty = \mathbf{N}$ if and only if $f|A = \mathbf{N}$ ($\mathbf{N}$ is the new multiplicative zero element anyway). Restricting the constant $\mathbf{N}$ yields $\mathbf{N}$, and so there is only one $f|A$ (i.e. $\mathbf{N}$) for which $\|f|A\|_\infty = \mathbf{N}$.

If $f \in \ell^\infty(S)$ then f is bounded, so any restriction is bounded (with at most the same bound as f), hence $f|A \in \ell^\infty_\mathbf{N}(S)$. Finally, to obtain any given $g \in \ell^\infty_\mathbf{N}(S)$ we can set

$$f(t) = \begin{cases} g(t) & \text{if } g(t) \neq \mathbf{N} \\ 0 & \text{if } g(t) = \mathbf{N} \end{cases}$$

for all t, so clearly $f \in \ell^\infty(S)$, and $A = \mathbf{d}(g)$, so thus $g = f|A$. $\square$

I shall now focus again on real-valued functions only.

Definition 6.28 Let $m : \ell^\infty_\mathbf{N}(S) \to \mathbb{R} \cup \{\mathbf{N}\}$. m is a *restrictable mean* if, for any $f|A \in \ell^\infty_\mathbf{N}(S)$,

$$\inf_{s \in A} f(s) \leq m(f|A) \leq \sup_{s \in A} f(s),$$

with $m(\mathbf{N}) := \mathbf{N}$, and

1. $m(f|A + g|A) = m((f+g)|A) = m(f|A) + m(g|A)$ for each $f, g \in \ell^\infty(S)$ and $A \subseteq S$, and

2. $\lambda m(f|A) = m(\lambda f|A)$ for any $\lambda \in \mathbb{R} \cup \{\mathbf{N}\}$.

Remark 6.29 Since $f|A$ refers to both a function f and its domain A, there is less need to give each individual function some label such as f. To write a statement or expression with less labelling, sometimes we will write a restricted function $f|A$ in the form $(t \mapsto f(t))|A$. For example, if f is given by $f(t) = t^2 + 3$, then $f|A = \left(t \mapsto t^2 + 3\right)|A$.

It follows from the above definitions that $m(f|\{x\}) = f(x)$, and

$$\min\{m(f|A), m(f|B)\} \leq m(f|A \cup B) \leq \max\{m(f|A), m(f|B)\}.$$

When operating on unrestricted bounded functions, such an m is therefore a mean in the usual sense, so we will write $m(f|S) = m(f)$.

Finally, we complete the connection between restricted means, restricted functions, and indicator functions, by noting that

$$m(f \cdot \hat{\chi}_A|B) = m(f \cdot \hat{\chi}_B|A) = m(f|A \cap B) = m(f \cdot \hat{\chi}_{A \cap B}) = m(f \cdot \hat{\chi}_A \cdot \hat{\chi}_B),$$

and with the constant function $\hat{\chi}_\emptyset = \mathbf{N}$,

$$m(\mathbf{N}) = m(f \cdot \hat{\chi}_\emptyset) = m(f|\emptyset) = \mathbf{N},$$

and, similarly to the norm, this is the only way to obtain $\mathbf{N}$ from a mean. Thus, for instance,

$$\hat{\chi}_A(t) = m(1|\{t\} \cap A) = m\left(\hat{\chi}_{\{t\}}|A\right),$$

and for $f \in \ell^\infty_{\mathbf{N}}(S)$,

$$m(f|S \backslash \mathbf{d}(f)) = \mathbf{N}.$$

Definition 6.30 Let m be a restrictable mean on $S \times S$ (that is $m : \ell^\infty_{\mathbf{N}}(S \times S) \to \mathbb{R} \cup \{\mathbf{N}\}$ with the usual properties). Define the *convolution of* f *and* g *with respect to* m, written $f *_m g$, by setting for all $u \in S$,

$$(f *_m g)(u) := m((s,t) \mapsto f(s)g(t)|\{(s,t) : st = u\}).$$

Lemma 6.31 For any $f, g \in \ell^\infty_{\mathbf{N}}(S)$ and restrictable mean m on $S \times S$, $(f *_m g) \in \ell^\infty_{\mathbf{N}}(S)$.

Proof For any point $u \in S$, $|(f *_m g)(u)| = |m(f(s)g(t)|st = u)|$ is, by definition of m, less than $\sup_{st=u} |f(s)g(t)| \leq \|f\|_\infty \|g\|_\infty$. $\square$

Now, let $s *_m f := \hat{\chi}_{\{s\}} *_m f$, and similarly $f *_m s := f *_m \hat{\chi}_{\{s\}}$. Since for any $s \in S, A \subseteq S$,

$$A \cap s^{-1}\{u\} \neq \emptyset \Leftrightarrow (\exists t)(st = u \text{ and } t \in A) \Leftrightarrow u \in sA$$

it—perhaps unsurprisingly—follows that

$$(s *_m \hat{\chi}_A)(u) = m\left(\hat{\chi}_A \middle| s^{-1}\{u\}\right) = \hat{\chi}_{sA}(u),$$

and in particular, $s *_m t \equiv \hat{\chi}_{\{s\}}\hat{\chi}_{\{t\}} = \hat{\chi}_{\{st\}} \equiv st$, again making $s \mapsto \hat{\chi}_{\{s\}}$ a faithful semigroup $\cdot$-$*_m$-homomorphism from S. This holds for any restrictable mean m.

* * *

So when do we get $m(s *_n f) = m(f)$? If we construct m out of some left-invariant μ, does it matter what we choose for n? Sometimes, but not in an obviously useful manner. Suppose s acts on A injectively, and consider $s *_n \chi_A$. If $\mu(A) = 1$ then we expect that $m(\chi_A) = 1$. Now, for certain $u \in S$, $n\left(\chi_A \middle| s^{-1}\{u\}\right) \leq \sup_{st=u} \chi_A(t) = 1$ by definition, so even though s acts injectively on A, $s *_n \chi_A \leq s \circledast \chi_A$, and thus $m(s *_n \chi_A) \leq m(\chi_A) = m(s \circledast \chi_A)$. In other words, while the formulation for $*_n$ provides a generalisation of convolution that always behaves sensibly in $\ell^\infty(S)$, it takes us further from thinking about amenability and fair amenability with means on $\ell^\infty(S)$.

Chapter 7

Making Other Conditions Fair

7.1 Preimage invariance

Consider the preimage of A under the action of s: $s^{-1}A := \{t : st \in A\}$. Here s^{-1} denotes the preimage of the left regular map λ_s, and does not necessarily correspond to an inverse element of s. However, the notation is not problematic, because if S is a group, then $\left\{s^{-1}a : a \in A\right\} = \{t : st \in A\}$. When interpreting classical amenability in terms of finitely-additive measures, they are considered to be *totally* invariant under the preimages. More precisely, a left-invariant mean gives rise to a finitely-additive measure μ *not* satisfying $\mu(A) = \mu(sA)$ but instead satisfying

$$\mu(A) = \mu\left(s^{-1}A\right)$$

for every $A \subseteq S$ and $s \in S$. This is equivalent to ordinary invariance when dealing with groups. As discussed previously, this *preimage invariance* is a fickle condition for semigroups with respect to the number of left zeroes. Can it be made to work?

We can talk about a number of sets that use preimages, for example, $ss^{-1}A = \{st : st \in A\}$ (elements of A divisible by s), and $s^{-1}sA = \{t : st \in sA\}$. It is easy to see that

$$ss^{-1}A \subseteq A \subseteq s^{-1}sA.$$

Now, $ss^{-1}A = A$ when everything in A is divisible by s. An example of when $s^{-1}sA = A$ is when $A = S$.

Also,

$$ss^{-1}sA = s\{t : st \in sA\} = \{st : st \in sA\} = sA$$

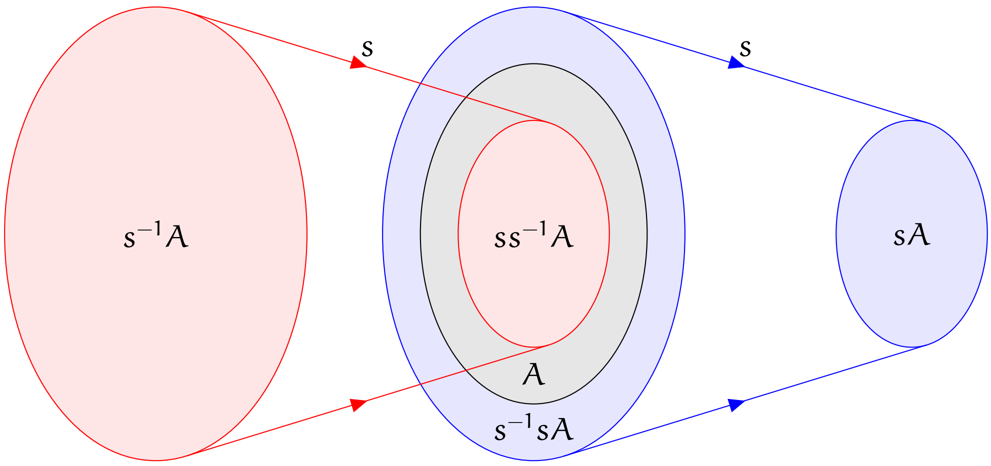

Figure 7.1: The sets $s^{-1}A$, A, and sA in relation to $s^{-1}sA$ and $ss^{-1}A$ when A is a general subset of S (containing both s-divisible and non-s-divisible elements).

and

$$s^{-1}ss^{-1}A = \{t : st \in \{su : su \in A\}\} = \{t : st \in A\} = s^{-1}A,$$

and so $ss^{-1}A = A$ when $A = sB$ for some B, which is another way of saying each element in A is left-divisible by s.

Consider the entire semigroup S: $s^{-1}S = \{t : st \in S\} = S$, so $sS = ss^{-1}S$. Thus we can think of the principal ideal sS as the set of all left s-divisible elements in S.

Perhaps there is an analogue of sub-invariance for preimages that looks like, for $s \in S$ and $A \subseteq S$, $\mu\left(s^{-1}A\right) \geq \mu(A)$. This seems reasonable because for a non-injective s, each $a \in A$ could be mapped to from more than one $x \in S$. The flaw here is that there could be none! Take, for instance, some suitably large set A containing no element divisible by s, so $s^{-1}A = \emptyset$. "$\mu(\emptyset) \geq \mu(A)$" doesn't seem so reasonable.

Fortunately we can ensure that $\left|s^{-1}\{a\}\right| \geq 1$ for all $a \in A$ by supposing that $ss^{-1}A = A$, or alternatively, that $A = sB \subseteq sS$ for some B. In this case, $\left|s^{-1}A\right| \geq |A|$ and so we get the following condition.

Definition 7.1 (Preimage sub-invariance) Let μ be a finitely-additive measure on the semigroup S. If for every $s \in S$ and $A \subseteq sS$,

$$\mu\left(s^{-1}A\right) \geq \mu(A)$$

then μ is left *preimage sub-invariant*.

Alternatively, since $ss^{-1}A = A \cap sS$ and over all $A \in \mathcal{P}(S)$ this accesses every subset of sS, we can use the condition

$$\mu\left(s^{-1}A\right) \geq \mu\left(ss^{-1}A\right)$$

for all A. This proves that

Lemma 7.2 If μ is left sub-invariant, then μ is left preimage sub-invariant.

Proof As above. $\square$

Now, we are interested in cases where $|A| = \left|s^{-1}A\right|$ for finite sets on either side, since this could be used in a similar manner. Suppose s and A satisfy

$$\forall a \in A : \exists x \in S : s^{-1}\{a\} = \{x\},$$

the emphasis being on the right-hand side, which is precisely a one-element set. This occurs when the "action" of s^{-1} is an (injective) *function* on A. It is easy to see that if s^{-1} behaves as function then it cannot be non-injective, otherwise the action of s is not a function. Each element of A is then *uniquely* left-divisible by s, and one consequence is that $\left|s^{-1}A\right| = |A|$.

Definition 7.3 (Fairly preimage invariant) Let μ be a finitely-additive measure on the semigroup S. If for every $s \in S$ and $A \subseteq sS$ with each element $a \in A$ uniquely left-divisible by s we have

$$\mu\left(s^{-1}A\right) = \mu(A)$$

then μ is *left fairly preimage invariant*.

How does this relate to fair invariance?

Lemma 7.4 The element s acts injectively on the left of $s^{-1}A$ if, and only if, every element of $ss^{-1}A$ is uniquely left-divisible by s.

Proof Suppose s acts injectively on $s^{-1}A$, and assume that there is some $a \in ss^{-1}A$ that is not uniquely left-divisible by s, i.e. that there exist distinct $u, v \in s^{-1}\{a\} = \{t : st = a\}$ (we need not be concerned with $s^{-1}\{a\}$ being empty). Thus $su = a = sv$. $s^{-1}\{a\} \subseteq s^{-1}A$, so by injectivity on $s^{-1}A$, $su = sv$ implies $u = v$, contradicting the assumption, and hence the injective left action of s gives us unique left-divisibility.

Conversely, suppose every $a \in ss^{-1}A$ is uniquely left-divisible. Thus if $su = a = sv$ then $u, v \in s^{-1}\{a\}$ and therefore $u = v$. But u, v belong in $s^{-1}A$ so the action of s is injective. □

Yet again, since $ss^{-1}A = A \cap sS$, we can replace in Definition 7.3 subsets A of sS with arbitrary A and the condition

$$\mu\left(s^{-1}A\right) = \mu\left(ss^{-1}A\right)$$

and thus it follows that

Lemma 7.5 If μ is a finitely-additive fairly invariant measure on S, then it is also preimage invariant.

Proof As described above. □

Thus we have defined a reasonable-looking weakening of fair amenability. When are the two equivalent? Perhaps if S is simple: in this case, every element divides every other, so Definition 7.3 applies to every set. But in order to reuse the same μ, one must have every injective act arising from unique divisibility. Cancellativity is not sufficient either, for the opposite reason: every injective act corresponds to unique divisibility, but not every set is contained in the required ideal. It is easy to see how everything coincides nicely when dealing with simple cancellative semigroups (i.e. groups).

Recall that for any s and A, there exists some maximal $B \subseteq A$ such that $sA = sB$ and $s \cdot B$ is injective. Suppose A does not have the uniquely left-s-divisible property. There is not necessarily a way of restricting A to some subset B such that $s^{-1}A = s^{-1}B$ and B does have the property.

However, we can restrict $s^{-1}A$. What we mean is that any set $A = sB \subseteq sS$ is uniquely left-divisible by s *with respect to* some subset B' of $B = s^{-1}A$, or else uniqueness is lost, and whenever $s^{-1}\{a\}$ for $a \in A$ is not a singleton set, we can apply choice to obtain it. But as is evident from Figure 7.2, this is merely a restatement of fair invariance, which we wanted to imitate in the first place.

Question 7.6 So which semigroups have finitely-additive preimage-invariant probability measures, outside those that are fairly amenable?

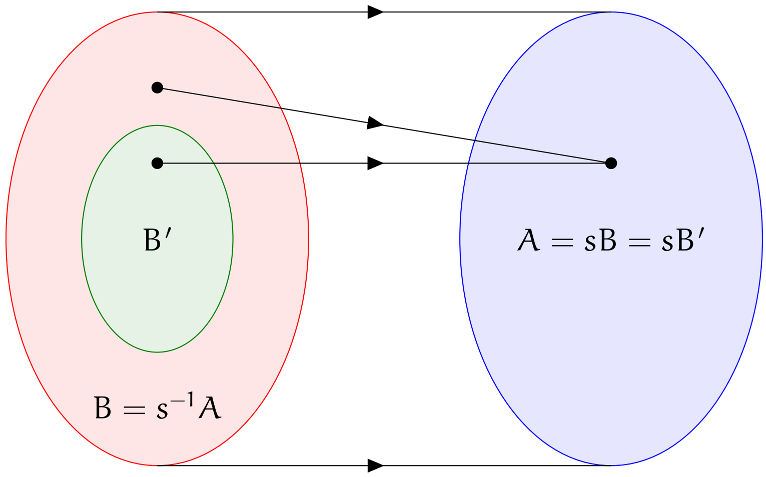

Figure 7.2: A maximal subset B' of B such that A is uniquely left s-divisible with respect to B…is merely the same as a subset B' of B such that s acts injectively on the left of B'.

7.2 Inner amenability

Definition 7.7 A group G is *inner amenable* if it supports a mean $m \in L^\infty(G)^*$ satisfying

$$m(f) = m\left(g^{-1}fg\right) \quad \text{for all } g \in G, f \in L^\infty(G).$$

Such a mean m is *inner invariant*. (Paterson, 1988)

For each $g \in G$ consider the map $\mathfrak{I}_g : f \mapsto g^{-1}fg$ for all $f \in \ell^\infty(G)$. The map $g \mapsto \mathfrak{I}_g$ is the *inner automorphism*, which is the mapping under which inner invariant means are invariant. This justifies the nomenclature. Inner amenability clearly follows very easily from amenability, and is trivial for Abelian groups.

However, *all* discrete groups are inner amenable. This is easily seen by the following. If $e \in G$ is the identity element, and $\delta_e \in \ell^\infty(G)^*$ is the *trivial mean* given by $\delta_e(f) = f(e)$ for all $f \in \ell^\infty(G)$, then it is easy to see that

$$\delta_e\left(g^{-1}fg\right) = f\left(g^{-1}eg\right) = f(e) = \delta_e(f),$$

for all $g \in G$. An obvious variation on inner amenability is to remove δ_e from consideration: a group G is said to be *trivially inner amenable* if δ_e is the only such inner invariant mean, and interest lies in finding inner, but not trivially inner, amenable groups. The investigation into inner invariant means was initiated by Effros (1975), who showed that all Property Γ groups are inner amenable but not merely trivially so. There are, however, non-discrete locally-compact groups which fail to be trivially inner amenable, and so sometimes the original definition is preferable. Another class of groups that are inner amenable are the $[IN]$-groups (Lau and Paterson, 1991).

An analogue of the above definition for semigroups is given below.

Definition 7.8 A semigroup S is *inner amenable* if it supports a mean $m \in L^{\infty}(S)^{*}$ satisfying

$$m(sf) = m(fs) \quad \text{for all } s \in S, f \in L^{\infty}(S).$$

Such a mean m is *inner invariant.* (Ling, 1997)

Semigroup inner amenability is certainly a weaker condition than amenability, which is itself quite weak already. If S is a monoid, then δ_1 is again a inner invariant mean:

$$\delta_1(sf) = f(s1) = f(s) = f(1s) = \delta_1(fs).$$

If S has a zero element, then $\delta_0 : f \mapsto f(0)$ is also inner invariant:

$$\delta_0(sf) = f(s0) = f(0) = f(0s) = \delta_0(fs).$$

Conceivably, many such "trivially" inner invariant means would need to be stamped out to create a suitably interesting condition. By analogy with fair amenability, it may be worthwhile starting with a condition that is too strong, and then weaken it. There may be inspiration in the following: if S is a semigroup with inner invariant mean m, then the finitely-additive measure μ defined by $\mu(A) = m(\chi_A)$ satisfies

$$\mu\left(s^{-1}A\right) = \mu\left(As^{-1}\right) \quad \text{for all } s \in S, A \subseteq S.$$

7.2.1 Inner $\circledast$- and $*$-invariance

Definition 7.9 Let S be a semigroup, $\mu \in [0,1]^{\mathcal{P}(S)}$ a finitely-additive measure.

(i) μ is *inner $\circledast$-invariant* if

$$\mu(sA) = \mu(As) \quad \text{for all } A \subseteq S, s \in S.$$

(ii) μ is *inner $*$-invariant* if $\mu(sA) = \mu(As)$ merely for those $s \in S$ and $A \subseteq S$ such that s acts injectively on both the left and right of A.

Definition 7.10 Let S be a semigroup, $m \in \ell^\infty(S)^*$ a mean.

(i) m is *inner* $*$*-invariant* if

$$m(s * f) = m(f * s)$$

for all $s \in S$ and $f \in \ell^\infty(S)$ such that both $s * f$ and $f * s$ are in $\ell^\infty(S)$.

(ii) m is *inner* $\circledast$*-invariant* if

$$m(s \circledast f) = m(f \circledast s)$$

for all $s \in S$ and $f \in \ell^\infty(S)$ (both sides always exist in $\ell^\infty(S)$).

The main result will be the equivalence of inner $\circledast$-invariant means with inner $\circledast$-invariant measures, and similarly, inner $*$-invariant means and measures. But first, some observations.

Proposition 7.11 Inner $\circledast$-amenability is trivial for Abelian semigroups and groups (as $sA = As$ for all s and A), and for semigroups and groups supporting a totally-invariant finitely-additive probabiliy measure, in particular, the amenable groups. Additionally, adjoining an identity or a zero does not affect whether a semigroup is inner $*$-amenable, since $1A = A = A1$ and $0A = \{0\} = A0$ for all sets $A \subseteq S$.

Inner $*$-amenability follows trivially from inner $\circledast$-amenablility, but also from fair amenability.

In light of Proposition 7.11, the definition of inner $\circledast$-amenable seems to attain a good compromise, with no further weakening (e.g. to inner $*$-amenability) required.

Theorem 7.12 A semigroup S supports an inner $\circledast$-invariant finitely-additive measure if, and only if, it supports an inner $\circledast$-invariant mean.

Proof Let μ be an inner $\circledast$-invariant finitely-additive measure for S. Then define $m \in \ell^\infty(S)$ by setting

$$m(f) := \int f d\mu \quad \text{for all } f \in \ell^\infty(S).$$

By similar working to Lemma 6.19 and successive results, it boils down to using

$\mu(sA) = \mu(As)$ on the basis sets of the simple functions, and therefore

$$m(s \circledast f) = \int (s \circledast f)d\mu = \int (f \circledast s)d\mu = m(f \circledast s)$$

for all $f \in \ell^\infty(S)$ and $s \in S$, as required.

Conversely, let m be an inner $\circledast$-invariant mean. Since $s \circledast \chi_A = \chi_{sA}$ for all A, and similarly on the right, we can set μ to be a measure given by

$$\mu(A) := m(\chi_A) \quad \text{for all } A \subseteq S,$$

and therefore,

$$\mu(sA) = m(s \circledast \chi_A) = m(\chi_A \circledast s) = \mu(As),$$

for all $s \in S$ and $A \subseteq S$, as required.

The next few results show the equivalence of the inner $*$-invariant measures and means.

Lemma 7.13 Let S be a semigroup with an inner $*$-invariant measure μ. Then

$$\int (s * \chi_A)d\mu = \int (\chi_A * s)d\mu$$

for all $s \in S$ and $A \subseteq S$ such that $s * \chi_A$ and $\chi_A * s$ are both in $\ell^\infty(S)$.

Proof By Lemma 6.2, there exists two finite partitions of A, $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$, such that s acts injectively on the left of each A_i and injectively on the right of each B_j, and in particular, $s*\chi_{A_i} = \chi_{sA_i}$ and $\chi_{B_j} * s = \chi_{B_j s}$. Let $C_{ij} := A_i \cap B_j$for each $i \in I, j \in J$. Thus $\{C_{ij}\}_{i,j}$ is a finite partition (into $|I \times J|$ sets) of A such that s acts injectively on the left and the right of each C_{ij}. Then, in a similar vein to Lemma 6.4,

$$\begin{aligned}
\int (s * \chi_A) d\mu &= \int \left(s * \sum_{(i,j)\in I\times J} \chi_{C_{ij}} \right) d\mu \\
&= \sum_{(i,j)\in I\times J} \int \chi_{sC_{ij}} d\mu \\
&= \sum_{(i,j)\in I\times J} \mu(sC_{ij}) = \sum_{(i,j)\in I\times J} \mu(C_{ij}s) \\
&= \sum_{(i,j)\in I\times J} \int \chi_{C_{ij}s} d\mu \\
&= \int (\chi_A * s) d\mu,
\end{aligned}$$

as required. □

Remark 7.14 Lemma 7.13 extends, via similar working to Lemmas 6.5, 6.6, and 6.7, to show that if both $s * f$ and $f * s \in \ell^\infty(S)$, then $\int (s * f) d\mu = \int (f * s) d\mu$.

Thus the integral with respect to μ again suffices as an invariant mean. This final result shows the converse.

Lemma 7.15 Let S be a semigroup supporting an inner $*$-invariant mean $m \in \ell^\infty(S)^*$. Then there is an inner $*$-invariant finitely-additive measure μ.

Proof Let $\mu \in [0,1]^{\mathcal{P}(S)}$ be given by $\mu(A) := m(\chi_A)$ for all $A \subseteq S$. If s acts injectively on the left and right of A, then $s * \chi_A = \chi_{sA}$ and $\chi_A * s = \chi_{As}$, and so

$$\mu(sA) = m(s * \chi_A) = m(\chi_A * s) = \mu(As),$$

as required. □

It would be interesting to see how much further this idea could be stretched.

7.3 Results yet to be shown

Some desirable results from classical amenability theory on semigroups are as follows.

(i) If a semigroup has a left invariant mean and a right invariant mean, then it has a bi-invariant mean (Theorem 4.14);

(ii) For an amenable semigroup, there exists for every $x \in S$, $\epsilon > 0$, an ϵ-invariant finite mean (Theorems 2.11 and 4.13).

(iii) As a consequence of the existence of almost-invariant finite means, some Følner-type conditions (e.g. WFC) are necessary for amenable semigroups.

(iv) Every Abelian semigroup is amenable (Theorem 4.17).

These are important results, so it is worth asking what makes proving analogues of these for fair amenability difficult.

7.3.1 Bi-$*$-invariant means?

The definition of bi-$*$-invariant means is clear: a mean m is bi-$*$-invariant if it is both left $*$-invariant and right $*$-invariant. If S is a monoid with identity element 1, then this can be shortened to the requirement that

$$m(\phi) = m(s * \phi * t)$$

for all $s, t \in S$ and $\phi \in \ell^\infty(S)$ such that $s{*}\phi{*}t \in \ell^\infty(S)$. (Clearly $1{*}\phi = \phi = \phi{*}1$.)

If instead both $s * \phi, \phi * t \in \ell^\infty(S)$, by associativity, therefore, $s * \phi * t \in \ell^\infty(S)$. This would suggest, perhaps, that combining left and right $*$-invariant means is possible.

Thus, suppose a semigroup S supports a left $*$-invariant mean m and a right $*$-invariant mean n. Can we construct a bi-$*$-invariant mean out of them? Theorem 4.14 relied on the Arens product $\odot$ of the left and right-invariant means having the desired property. Applying the same ideas to $*$, from the bottom-up we "define a product" $m \circledcirc n$ for two means m and n.

(i) The partial actions $s * \phi$ and $\phi * s$ are defined.

(ii) Define a product $\circledcirc$ between $\phi \in \ell^\infty(S)$ and a mean n by setting

$$\{\phi \circledcirc n\}(s) := n(s * \phi) \quad \text{for } s \in S.$$

(iii) Finally, for two means m and n, define

$$\{m \circledcirc n\}(\phi) := m(\phi \circledcirc n).$$

Step (ii) only really provides a partial function $\phi \circledcirc n$. This maps $s \mapsto n(s * \phi)$, but is partial since $s * \phi \in \ell^\infty(S)$ only for some s and some ϕ. This is a problem: both m and n are not necessarily defined on partial functions. This might be fixable using ideas from § 6.5.2.

7.3.2 Almost fairly invariant finite means?

There is the possibility of a version of Theorem 2.11 for fair amenability. What does the analogous condition look like?

Since the proof of Theorem 2.11 was given via contradiction, let us recall the proof in reverse. To obtain the contradiction to the assumption that is S left fairly amenable, we must extract some $r \in \ell^\infty(S)$ and $x \in S$ such that $x * r \in \ell^\infty(S)$

$$m(x * r) - m(r) \neq 0.$$

In the original proof this followed as a result of $\langle x * r - r, t\rangle \geq 1$ for all $t \in S$, which in turn was a specialisation of $\langle x * r - r, \nu\rangle \geq 1$, for all finite means $\nu \in \Phi(S)$. Obviously $\chi_t \in \Phi(S)$ for all $t \in S$. That it is true for all finite means is, from this perspective, merely an artefact of requiring a convex set for the Hahn-Banach theorem.

Thus a crucial part of the original proof of the theorem was showing that there was some $r \in \ell^\infty(S)$ such that

$$1 \leq \langle x * \nu - \nu, r\rangle = \langle \nu, x \cdot r - r\rangle \quad \text{for all } \nu \in \Phi(S).$$

Now, however, the $*$ action must apply to the $r \in \ell^\infty(S)$, therefore $*$ and its dual action must be exchanged. Thus the proof requires demonstrating there is a $r \in \ell^\infty(S)$ such that $x * r \in \ell^\infty(S)$ and that

$$1 \leq \langle x \cdot \nu - \nu, r\rangle = \langle \nu, x * r - r\rangle \quad \text{for all } \nu \in \Phi(S).$$

This has the following problems.

Firstly, $\|x \cdot \nu\|_1$ is not necessarily finite, and therefore the dual "action" is really a partial action when restricted to $\ell^1(S)$. This is less intuitively problematic than the case of $*$ and $\ell^\infty(S)$, since in every case, $x \cdot \nu$ is well-defined, and $x \cdot \nu \in \ell^\infty(S)$. For example, again suppose that S is infinite and contains a zero. Then if $\nu = \chi_{\{0\}}$ and $x = 0$, then $x \cdot \nu = \chi_S$. Thus some characterisation of $\cdot$ as a partial action, within $\ell^1(S)$ and $\Phi(S)$, is required.

Lemma 7.16 Let $\nu \in \ell^1(S)$, $\nu \geq 0$, and $x \in S$. Then $x \cdot \nu \in \ell^1(S)$ if, and only if, $\left|x^{-1}\operatorname{supp}(\nu)\right| < \infty$. In particular, for all $\nu \in \Phi(S)$, $x \cdot \nu \in \Phi(S)$ if, and only if, x acts injectively on the left of $\operatorname{supp}(\nu)$.

Proof

$$
\begin{aligned}
\|x \cdot \nu\|_1 &= \sum_{t \in S} \nu(xt) \\
&= \sum_{u \in \operatorname{supp}(\nu)} \left|x^{-1}u\right| \nu(u)
\end{aligned}
$$

which is finite if, and only if, $\left|x^{-1}u\right|$ is finite for all $u \in \operatorname{supp}(\nu)$, if, and only if, $\left|x^{-1}\operatorname{supp}(\nu)\right|$ is finite. $\square$

Lemma 7.17 Let $s \in S$ and $\phi \in \ell^\infty(S)$. $s * \phi \in \ell^\infty(S)$ if, and only if, there exists a finite bound B such that

$$
|\langle \phi, s \cdot \nu \rangle| \leq B \quad \text{for all } \nu \in \Phi(sS).
$$

Proof In either case, $\langle \phi, s \cdot \nu \rangle = \langle s * \phi, \nu \rangle$ by the usual duality.

Suppose $s * \phi \in \ell^\infty(S)$. As previously noted, $\operatorname{supp}(s * \phi) \subseteq sS$, so only finite means on sS need be considered. Then

$$
\begin{aligned}
|\langle \phi, s \cdot \nu \rangle| &= |\langle s * \phi, \nu \rangle| \\
&\leq \|s * \phi\|_\infty \|\nu\|_1 \\
&= \|s * \phi\|_\infty \quad \because \|\nu\|_1 = 1.
\end{aligned}
$$

Thus use $B = \|s * \phi\|_\infty$.

Conversely, suppose there exists such a B. For all $t \in sS$,

$$\begin{aligned} |\{s * \phi\}(t)| &= |\langle s * \phi, t\rangle| \\ &= |\langle \phi, s \cdot \nu\rangle| \quad \text{specialising } \nu = \chi_t \\ &\le B \quad \text{by hypothesis.} \end{aligned}$$

Hence $\|s * \phi\|_\infty \le B$, and $s * \phi \in \ell^\infty(S)$. □

7.3.3 Fairly amenable Abelian semigroups?

Recall that all Abelian semigroups are amenable, a fact which is an easy consequence of the Markov-Kakutani fixed-point theorem. Results from Chapter 5 are almost enough to show the same for fairly amenable semigroups. An Abelian semigroup is left fairly amenable if, and only if, it is right fairly amenable (Lemma 5.43), so we only have to worry about one side. Abelian semigroups are directed unions of their finitely-generated subsemigroups, so we can use Theorem 5.47 to reduce the problem to the finitely-generated Abelian semigroups. Of those, all finite semigroups and the free Abelian semigroups on n generators are known examples, but these might not be the only finitely-generated Abelian semigroups. Some work remains to fill in the gaps. Therefore there is interest in applying the Markov-Kakutani theorem.

The fixed points are the means m invariant under the action of s, which is given by

$$\{s \cdot m\}(f) := m(s \cdot f) \quad \text{for all } f \in \ell^\infty(S), s \in S.$$

Given a mean m, what should be $s * m$? A reasonable suggestion is to set

$$\{s * m\}(f) := m(s * f) \quad \text{for all } f \in \ell^\infty(S), s \in S.$$

This defines $s * m$ only on $\ell^{s*}(S)$, so $s * m \in \ell^{s*}(S)^*$. (This is less problematic than the $\phi \circledcirc n$ introduced earlier.) The next logical question: Is $s*$, in some sense, an affine map on the set of means? This follows from being affine, in some sense, on the bounded functions.

Lemma 7.18 On $\ell^\infty(S)$ the partial map $f \mapsto s * f$ is affine in the sense that for all $f_1, f_2 \in \ell^\infty(S)$ such that $s * f_1, s * f_2 \in \ell^\infty(S)$,

$$s * (tf_1 + (1-t)f_2) = t(s * f_1) + (1-t)(s * f_2) \quad \text{for all } t \in [0, 1].$$

Proof Straightforward: if $s * f \in \ell^\infty(S)$ then $\alpha(s * f) = s * \alpha f \in \ell^\infty(S)$ for all $\alpha \in \mathbb{C}$. Similarly, if $s * f_1, s * f_2 \in \ell^\infty(S)$ then $a_1(s * f_1) + a_2(s * f_2) \in \ell^\infty(S)$ for all $a_1, a_2 \in \mathbb{C}$. Then

$$\begin{aligned} t(s * f_1) + (1-t)(s * f_2) &= s * tf_1 + s * (1-t)f_2 \\ &= s * (tf_1 + (1-t)f_2)\,, \end{aligned}$$

as required. □

Does the Markov-Kakutani theorem apply directly to demonstrate an $*$-invariant mean? No. The assumption of the theorem is that the commuting affine maps form an Abelian semigroup, and while the partial maps $m \mapsto s * m$ are continuous and affine as demonstrated above, they would instead form a semi-category. Proof of the Markov-Kakutani theorem makes use of the fact that the repeatedly iterated application of each particular affine map is defined, but there is no guarantee that $s^n * m$ is defined for all n. Nevertheless there may be some generalisation of the Markov-Kakutani theorem that could be used.

7.3.4 Other outstanding questions

Some of the questions that were raised but not resolved, and a few extras, are summarised here.

Interesting classes of semigroups where the fair amenability is not yet understood, or not completely characterised, are:

(i) Abelian semigroups in general (c.f. Propositions 5.48, 5.49),

(ii) Inverse semigroups, in particular free inverse semigroups with more than one generator (c.f. Example 5.60),

(iii) Semilattices,

(iv) Completely regular semigroups (c.f. §5.2.7), and

(v) Graph inverse semigroups, and the inverse semigroups of left cancellative Leech categories (Remark 5.56).

Other questions include:

(i) How exactly would the theory of fair amenability be extended to account for topology (c.f. §2.7)?

(ii) In light of current research into partial actions on C*-algebras, what are the properties of the $*$-partial action, and how does it fit into the theory? A survey of partial actions was given by Dokuchaev (2011).

(iii) Can the Mitsch partial order be used to generalise the left regular representation of an inverse semigroup on a Hilbert space to arbitrary semigroups, and hence the weak containment property?

(iv) How exactly does fair amenability relate to growth rates and conditions, and other geometric conditions?

(v) There is an apparent divide between cases with one generator, which seem to be quite easily shown to be fairly amenable, and cases with two or more generators, which tend not to be fairly amenable at all. Is it possible to refine the boundary between the two? One possible avenue here may be to investigate the graph inverse semigroups.

Bibliography

Adian, S I (1983). “Random walks on free periodic groups”. In: *Math. U.S.S.R. Izvestiya* 21, pp. 425–434.

Agnew, R P and A P Morse (1938). “Extensions of linear functionals, with applications to limits, integrals, measures, and densities”. In: *The Annals of Mathematics* 39.1, pp. 20–30.

Akhmedov, Azer (2009). “A new metric criterion for non-amenability III: Non-amenability of R.Thompson’s group F”. In: *arXiv.org*, pp. 1–53.

Akhmedov, Azer et al. (2009). “On Shavgulidze’s Proof of the Amenability of some Discrete Groups of Homeomorphisms of the Unit Interval”.

Allan, Graham R (2011). *Introduction to Banach Spaces and Algebras*. Oxford Graduate Texts in Mathematics. Oxford University Press.

Argabright, L N and C O Wilde (1967). “Semigroups satisfying a strong Følner condition”. In: *Proceedings of the American Mathematical Society* 18.4, pp. 587–591.

Barnes, Bruce A (1976). “Representations of the ℓ^1-algebra of an inverse semigroup”. In: *Transactions of the American Mathematical Society* 218, pp. 361–396.

Bergelson, Vitaly (2005). “Multiplicatively large sets and ergodic Ramsey theory”. In: *Israel Journal of Mathematics* 148.1, pp. 23–40.

Bleak, Collin and Bronlyn Wassink (2007). “Finite index subgroups of R. Thompson’s group F”. In: *arXiv.org* x, pp. 1–26.

Burillo, José (1999). “Quasi-Isometrically Embedded Subgroups of Thompson’s Group F”. In: *Journal of Algebra* 212.1, pp. 65–78.

Burillo, José, Sean Cleary, and Bert Wiest (2007). “Computational explorations in Thompson’s group F”. In: *Geometric Group Theory*, pp. 21–35.

Burton, R M and M Keane (1989). “Density and Uniqueness in Percolation”. In: *Communications in Mathematical Physics*, pp. 501–505.

Calegari, Danny (July 2009). *Amenability of Thompson's group F?* URL: `http://lamington.wordpress.com/2009/07/06/amenability-of-thompsons-group-f/`.

Cannon, J W, W J Floyd, and W R Parry (1994). "Introductory Notes on Richard Thompson's groups". In: *Research Report GCG63, The Geometry Center, Univ. Minn*.

Ceccherini-Silberstein, Tullio and Michel Coornaert (Aug. 2010). *Cellular Automata and Groups*. Springer.

Clifford, A H and G B Preston (1967). *The Algebraic Theory of Semigroups*. Vol. 2. American Mathematical Society Mathematical Surveys. American Mathematical Society.

Day, Mahlon M (1942). "Ergodic theorems for abelian semi-groups". In: *Transactions of the American Mathematical Society* 51.2, pp. 399–412.

— (1957). "Amenable semigroups". In: *Illinois Journal of Mathematics*.

— (1969). "Semigroups and Amenability". In: *Semigroups / Proceedings of a Symposium on Semigroups Held at Wayne State University, Detroit, Michigan, June 27-29, 1968*. Ed. by Karl W Folley. Academic Press, pp. 5–53.

— (June 1983). "This Week's Citation Classic". In: *Current Contents*, p. 18.

Dokuchaev, M (2011). "Partial actions: a survey". In: *Contemp. Math* 537, pp. 173–184.

Donnelly, John (2012). "Subsemigroups of Cancellative Amenable Semigroups". In: *Int. J. Contemp. Math. Sciences* 7.23, pp. 1131–1137.

Duncan, J and I Namioka (1978). "Amenability of inverse semigroups and their semigroup algebras". In: *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* 80, pp. 309–321.

Duncan, J and Alan LT Paterson (1985). "C*-algebras of inverse semigroups". In: *Proceedings of the Edinburgh Mathematical Society (Series 2)* 28.01, pp. 41–58.

Dunford, Nelson and Jacob T Schwartz (1958). *Linear Operators Part I: General Theory*. New York: Interscience Publishers, Inc.

Effros, Edward G (1975). "Property Γ and inner amenability". In: *Proceedings of the American Mathematical Society* 47.2, pp. 483–486.

Elder, Murray, Andrew Rechnitzer, and Thomas Wong (2011). "On the cogrowth of Thompson's group F". In: *Journal of Algebra (Computational Algebra) (preprint)*, pp. 1–21.

Elder, Murray et al. (Nov. 2013). “A Metropolis Markov Chain algorithm to sample trivial words and compute cogrowth in finitely generated groups; or - F is not amenable”. In: *Victorian Algebra Conference*.

Følner, Erling (1955). “On groups with full Banach mean value”. In: *Math. Scand* 3, pp. 243–254.

Grigorchuk, Rostislav I (1991). “On growth in group theory”. In: *Proceedings of the International Congress of Mathematicians*, pp. 325–338.

— (1996). “On a problem of day on nonelementary amenable groups in the class of finitely defined groups”. In: *Mathematical Notes* 60.5, pp. 580–582.

Gromov, Mikhael (1987). “Hyperbolic Groups”. In: *Essays in Group Theory*. Ed. by S M Gersten. New York: Springer Verlag, pp. 75–263.

Gunawardena, Jeremy (Mar. 2001). “An Introduction to Idempotency ”. In: pp. 1–50.

Hofstadter, Douglas R. (1979). *Gödel, Escher, Bach: An Eternal Golden Braid*. New York, NY, USA: Basic Books, Inc. ISBN: 0465026850.

Howie, J.M. (1995). *Fundamentals of Semigroup Theory*. LMS monographs. Clarendon Press. ISBN: 9780198511946. URL: `http://books.google.com.au/books?id=pPg6uuYuczoC`.

Howie, John Mackintosh (1976). *An Introduction to Semigroup Theory*. L.M.S. Monographs. Academic Press.

Janyska, Josef, Marco Modugno, and Raffaele Vitolo (Oct. 2007). “Semi–vector spaces and units of measurement”. In: *arXiv.org*. arXiv:`0710.1313v1 [math.AC]`.

Johnson, Barry Edward (1972). *Cohomology in Banach Algebras*. American Mathematical Soc.

Jones, David G and Mark V Lawson (2011). “Graph inverse semigroups: their characterization and completion”. In: *arXiv.org*, pp. 1–30.

Kaimanovich, Vadim A (2009). “Self-similarity and Random Walks”. In: *Fractal Geometry and Stochastics IV*. Ed. by Christoph Bandt, Peter Mörters, and Martina Zähle. Birkhäuser, pp. 1–292.

Kesten, Harry (1958). “Symmetric random walks on groups”. In: *Trans. Amer. Math. Soc* 1, pp. 336–354.

— (1959). “Full Banach mean values on countable groups”. In: *Mathematica Scandinavica* 7, pp. 146–156.

Klawe, Maria (1977). “Semidirect product of semigroups in relation to amenability, cancellation properties, and strong Folner conditions”. In: *Pacific Journal of Mathematics* 73.1, pp. 91–106.

Lamport, Leslie (2012). "How to write a 21st century proof". In: *Journal of Fixed Point Theory and Applications* 11.1, pp. 43–63.

Lau, Anthony To-Ming and Alan LT Paterson (1991). "Inner amenable locally compact groups". In: *Transactions of the American Mathematical Society* 325.1, pp. 155–169.

Lawson, Mark V (1998). *Inverse Semigroups*. World Scientific Publishing Co. P/L.

— (2004). "Representations of the Thompson group F via representations of the polycyclic monoid on two generators".

Ling, Joseph M (1997). "Inner amenable semigroups I". In: *Journal of the Mathematical Society of Japan* 49.3, pp. 603–616.

Luthar, Indar S (1962). "Uniqueness of the invariant mean on Abelian topological semigroups". In: *Transactions of the American Mathematical Society* 104.3, pp. 403–411.

Milan, David (2007). "C*-algebras of inverse semigroups: amenability and weak containment". In: *arXivorg* arXiv:math.

— (2008). "C*-algebras of inverse semigroups". PhD thesis. University of Nebraska.

Moore, Justin Tatch (2011). "Amenability and Ramsey Theory". In: *arXiv.org*, pp. 1–18.

— (2012). "Nonassociative ramsey theory and the amenability of thompson's group". In: *arXiv.org*, pp. 1–20.

Morse, Marston and Gustav A Hedlund (1944). "Unending chess, symbolic dynamics and a problem in semigroups". In: *Duke Mathematical Journal* 11.1, pp. 1–7.

Namioka, Isaac (1964). "Følner's Conditions For Amenable Semi-Groups". In: *Math. Scand* 15, pp. 18–28.

Ol'shanskii, Alexander Yu and Mark V Sapir (2002). "Non-amenable finitely presented torsion-by-cyclic groups". In: *Publications Mathématiques de l'Institut des Hautes Scientifiques* 96, 43–169 (2003).

On-Line Encyclopedia of Integer Sequences (2013). *A051662*. URL: `http://oeis.org/A051662`.

Paterson, Alan L T (1988). *Amenability*. American Mathematical Society.

— (1998). *Groupoids, Inverse Semigroups, and their Operator Algebras*. Boston: Birkhäuser.

Pavel, Liliana (2007). "Weak containment and hypergroup algebras". In: *Revue Roumaine de Mathematiques Pures et Appliquees* 52.1, pp. 87–94.

Peterson, Jesse (2011). *Topics in Analysis - Ergodic Theory (Math 390B)*. URL: http://www.math.vanderbilt.edu/~peters10/teaching/Spring2011/math390Blecturenotes.html.

Pier, Jean-Paul (Sept. 1984). *Amenable locally compact groups*. Wiley-Interscience.

Runde, Volker (2002). *Lectures on Amenability*. Lecture Notes in Mathematics. Springer Verlag.

Shavgulidze, E T (2009). *About amenability of subgroups of the group of diffeomorphisms of the interval.* Tech. rep. Moscow.

Tao, Terence (Apr. 2009). *Some notes on amenability*. URL: http://terrytao.wordpress.com/2009/04/14/some-notes-on-amenability/.

— (Jan. 2010). *Ultralimit analysis, and quantitative algebraic geometry*. URL: http://terrytao.wordpress.com/2010/01/30/the-ultralimit-argument-and-quantitative-algebraic-geometry/.

Tits, Jacques (1972). "Free subgroups in linear groups". In: *Journal of Algebra* 20.2, pp. 250–270.

van Douwen, Eric K (1992). "Finitely additive measures on $\mathbb{N}$". In: *Topology and its Applications* 47.3, pp. 223–268.

von Neumann, J (1929). "Zur allgemeinen Theorie des Masses". In: *Fundamenta Mathematica* 13, pp. 73–116.

Wagon, Stan (Sept. 1993). *The Banach-Tarski Paradox*. Cambridge University Press.

Wirzenius, Henrik (Aug. 2012). "Amenable Banach Algebras and Johnson's Theorem for the Group Algebra $L^1(G)$". MA thesis. University of Helsinki.

Yang, Zhuocheng (1987). "Følner numbers and Følner type conditions for amenable semigroups". In: *Illinois Journal of Mathematics* 31.3, pp. 496–517.

Index of Symbols

$\lvert\cdot\rvert$	Absolute value, or set cardinality.
$\dot\cup$	Disjoint union.
$\otimes$	Tensor product (of points, spaces, etc).
$\check\otimes$	Injective tensor product.
$\hat\otimes$	Projective tensor product.
$\triangle$	Set symmetric difference, i.e. $A\triangle B = (A\backslash B) \cup (B\backslash A)$
$\mathfrak{A}$	A typical Banach algebra.
(AC)	The Axiom of Choice is used in the proof.
$\mathcal{B}^1(\mathfrak{A}, E)$	The inner derivations from $\mathfrak{A}$ to E.
$\mathcal{B}(E, F)$	The space of bounded linear operators from E to F.
$\mathcal{B}(\mathcal{H})$	The C*-algebra of bounded linear operators on the Hilbert space $\mathcal{H}$.
$\mathbb{C}(S)$	The subalgebra of $\ell^1(S)$ consisiting of *finitely*-supported functions.
$C^*(S)$	The C*-algebra of the $*$-semigroup S.
$C^*_r(S)$	The reduced C*-algebra of $*$-semigroup S.
χ_A	The $\{0, 1\}$-valued indicator function for the set A: $\chi_A(x) = \lvert A \cap \{x\}\rvert$ for all x.
$\hat\chi_A$	The $\{\mathbf{N}, 1\}$-valued indicator function for the set A.
$\mathbf{d}(f)$	The domain of f.
∂A	The boundary of A.
E^*	The dual of E, i.e. $\mathcal{B}(E, \mathbb{K})$.
$\mathbb{F}_n$	The free group of rank n / having n generators.
$\mathbb{F}_\Sigma$	The free group on the alphabet Σ.

FS_n	The free semigroup on n generators.
FM_n	The free monoid on n generators.
FIS_n	The free inverse semigroup on n generators.
FIM_n	The free inverse monoid on n generators.
$f\|A$	Equal to the pointwise product $f \cdot \hat{\chi}_A$.
$\Gamma(G, S)$	Cayley graph of G with respect to generating set S.
$\mathcal{H}$	A typical Hilbert space.
$\mathcal{H}^1(\mathfrak{A}, E)$	The first Hochschild cohomology group of $\mathfrak{A}$ with coefficients in E.
$\mathbb{K}$	Either $\mathbb{R}$ or $\mathbb{C}$, as appropriate.
$\ell^1(S)$	The Banach ($*$-)algebra of absolutely-summable functions $S \to \mathbb{K}$ (usually with convolution).
$\ell^2(S)$	The Hilbert space of square-summable functions on S.
$\ell^\infty(S)$	The Banach space of bounded functions $S \to \mathbb{K}$.
$\ell^\infty_+(S)$	The Banach space of bounded non-negative functions $S \to \mathbb{R}_+$.
$\ell^{s*}(S)$	The linear space consisting of functions f satisfying $s * f \in \ell^\infty(S)$.
$\ell^\infty_{\mathbf{N}}(S)$	A linear space of bounded partial functions $S \mapsto \mathbb{K} \cup \{\mathbf{N}\}$.
λ_s	The left regular representation of s (as a self-map of the semigroup).
$\lim_{\mathcal{U}}$	An ultralimit in some fixed free ultrafilter $\mathcal{U}$ on $\mathbb{N}$.
$\mathfrak{M}(S)$	Subspace of $\ell^\infty(S)^*$ or $L^\infty(S)^*$ consisting only of means.
$\mathbf{N}$	A stand-in value interpreted as "not defined"; $f(x) = \mathbf{N}$ whenever $x \notin \mathbf{d}(f)$.
$\Phi(S)$	The set of finite means on S: $\nu \in \Phi(S)$ iff $\\|\nu\\|_1 = 1$, supp(ν) is finite, and $\nu \geq 0$.
P_n	The polycyclic monoid on n generators.
$P(S)$	Subspace of non-negative integrable functions from $L^1(S)$.
$P(S)^\wedge$	Embedding of $P(S)$ into $L^\infty(S)^*$ via the canonical embedding of $L^1(S)$ inside $L^\infty(S)^*$.
$PI(\mathcal{H})$	The inverse semigroup of partial isometries of $\mathcal{H}$.
$\pi_2(s)$	The left regular representation of s (as an $\ell^2(S)$ operator).

$\mathcal{P}(S)$	The power set of S, i.e. the Boolean algebra of all subsets of S.
S^0	The semigroup S with a zero element adjoined if necessary.
S^1	The semigroup S with an identity element adjoined if necessary.
Σ^+	The string semigroup on the alphabet Σ (note that $\Sigma^+ \cong FS_{\|\Sigma\|}$).
Σ^*	The string monoid on the alphabet Σ ($\Sigma^* \cong (\Sigma^+)^1$).
$\mathrm{supp}(f)$	The support of f, being the set $\{s : f(s) \neq 0\}$.
$\mathbb{T}$	The unit circle group $\left\{e^{i\theta} : \theta \in (-\pi, \pi]\right\}$.
$\mathcal{U}(\mathcal{H})$	The unitary group of the Hilbert space $\mathcal{H}$.
(UL)	The Ultrafilter Lemma is used in the proof.
Y^X	The set of functions mapping $X \to Y$.
$\mathcal{Z}^1(\mathfrak{A}, E)$	The derivations from $\mathfrak{A}$ to E.